

KHOVANOV HOMOLOGY, OPEN BOOKS, AND TIGHT CONTACT STRUCTURES

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ABSTRACT. We define the reduced Khovanov homology of an open book (S, ϕ) , and we identify a distinguished “contact element” in this group which may be used to establish the tightness or non-fillability of contact structures compatible with (S, ϕ) . Our construction generalizes the relationship between the reduced Khovanov homology of a link and the Heegaard Floer homology of its branched double cover. As an application, we give combinatorial proofs of tightness for several contact structures which are not Stein-fillable. Lastly, we investigate a comultiplication structure on the reduced Khovanov homology of an open book which parallels the comultiplication on Heegaard Floer homology defined in [5].

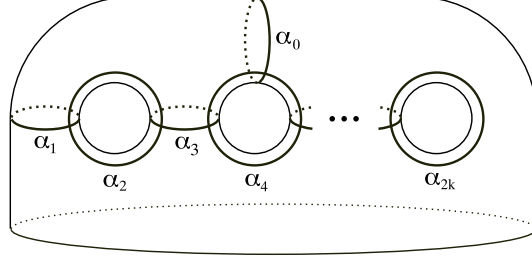
1. INTRODUCTION

The goal of this paper is to demonstrate how Khovanov homology and related ideas may be used to combinatorially establish the tightness or non-fillability of certain contact structures. Let S be a compact, oriented surface with boundary, and let ϕ be a composition of Dehn twists around homotopically non-trivial curves in S . The abstract open book (S, ϕ) corresponds to a contact 3-manifold, which we denote by $(M_{S, \phi}, \xi_{S, \phi})$ [10, 37]. In this paper, we use the link surgeries spectral sequence machinery of Ozsváth and Szabó [30] to define a filtered chain complex $(C(S, \phi), D)$ whose homology is isomorphic to $\widehat{HF}(-M_{S, \phi})$ (we work with \mathbb{Z}_2 coefficients throughout). We then define the *reduced Khovanov homology* of the open book (S, ϕ) to be the E^2 term of the spectral sequence associated to this filtered complex, and we identify an element $\psi(S, \phi) \in \widehat{Kh}(S, \phi)$ which is related to the Ozsváth-Szabó contact invariant $c(S, \phi) \in \widehat{HF}(-M_{S, \phi})$ via this spectral sequence.

Let $S_{k, r}$ denote the genus k surface with r boundary components. By Giroux’s correspondence [10], every contact 3-manifold is compatible with an open book of the form $(S_{k, 1}, \phi)$. Moreover, any boundary-fixing diffeomorphism of $S_{k, 1}$ is isotopic (rel. $\partial S_{k, 1}$) to a composition of Dehn twists around the curves $\alpha_0, \dots, \alpha_{2k}$ depicted in Figure 1 [15]. We show that $\widehat{Kh}(S_{k, 1}, \phi)$ and $\psi(S_{k, 1}, \phi)$ are combinatorially computable when ϕ is such a composition.

When the contact manifold $(M_{S, \phi}, \xi_{S, \phi})$ is the branched double cover of a transverse link, our construction specializes to the setup of [32]. Indeed, let $w = w(\sigma_1, \sigma_1^{-1}, \dots, \sigma_{2k}, \sigma_{2k}^{-1})$ be a word in the elementary generators (and their inverses) of the braid group on $2k+1$ strands, and let L_w denote the closure of the braid specified by w . We may think of L_w as a transverse link in standard contact structure ξ_{st} on S^3 , and lift ξ_{st} to a contact structure ξ_{L_w} on the double cover $\Sigma(L_w)$ of S^3 branched along L_w . The contact structure ξ_{L_w} is compatible with a natural open book decomposition $(S_{k, 1}, \phi_w)$ of $\Sigma(L_w)$, where $\phi_w = w(D_{\alpha_1}, D_{\alpha_1}^{-1}, \dots, D_{\alpha_{2k}}, D_{\alpha_{2k}}^{-1})$ (here, D_γ

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FIGURE 1. The surface $S_{k,1}$, and the curves $\alpha_0, \dots, \alpha_{2k}$.

stands for the right-handed Dehn twist around the curve γ). According to [30], the Heegaard Floer homology $\widehat{HF}(-\Sigma(L_w))$ can be computed via a link surgeries spectral sequence whose E^2 term is isomorphic to the reduced Khovanov homology of L_w . In this case, $\widehat{Kh}(S_{k,1}, \phi_w)$ is the same as $\widehat{Kh}(L_w)$, and the element $\psi(S_{k,1}, \phi_w)$ coincides with the transverse link invariant $\psi(L_w)$ introduced by the second author in [32]. As shown by L. Roberts in [35], $\psi(L_w)$ “corresponds” (in a sense to be made precise) to the contact invariant $c(\xi_{L_w}) \in \widehat{HF}(-\Sigma(L_w))$. Our construction generalizes this result.

The vector space $\widehat{Kh}(S, \phi)$ inherits a grading from the filtration of $(C(S, \phi), D)$, which we refer to as the “homological grading” (and also as the “ I -grading”) since it agrees with the homological grading on reduced Khovanov homology under the specialization described in the previous paragraph. The element $\psi(S, \phi)$ is contained in homological grading 0. In Section 3, we show that the graded vector space $\widehat{Kh}(S, \phi)$ is invariant under stabilization of the open book. The element $\psi(S, \phi) \in \widehat{Kh}(S, \phi)$ is invariant under positive stabilization, and is killed by negative stabilization. On the other hand, $\widehat{Kh}(S, \phi)$ is not an invariant of the isotopy class of ϕ (the rank of $\widehat{Kh}(S, \phi)$ depends on the precise way that ϕ is written as a composition of Dehn twists). Even so, the element $\psi(S, \phi)$ may sometimes be used, per the following theorem, to determine whether the contact structure $\xi_{S, \phi}$ is tight.

Theorem 1.1. *If the spectral sequence from $\widehat{Kh}(S, \phi)$ to $\widehat{HF}(-M_{S, \phi})$ collapses at the E^2 term then $\psi(S, \phi) \neq 0$ implies that $c(S, \phi) \neq 0$, and, hence, that $\xi_{S, \phi}$ is tight.*

Observe that this spectral sequence collapses at the E^2 term whenever

$$\text{rk}(\widehat{Kh}(S, \phi)) = |H_1(-M_{S, \phi}; \mathbb{Z})|,$$

since $\text{rk}(\widehat{HF}(-M_{S, \phi})) \geq |H_1(-M_{S, \phi}; \mathbb{Z})|$ [27]. Thus, in favorable cases, the collapsing condition is easy to verify (we use the computer programs **Kh** and **Trans** [3] to compute $\text{rk}(\widehat{Kh}(S, \phi))$ and to determine whether $\psi(S, \phi) \neq 0$). In the special case of branched double covers, we can often do without a computer since the spectral sequence from $\widehat{Kh}(L)$ to $\widehat{HF}(-\Sigma(L))$ collapses at the E^2 term as long as L is a quasi-alternating link [30]. In particular, if the transverse knot K belongs to a quasi-alternating knot type, then Theorem 1.1 implies that $c(\xi_K) \neq 0$ whenever $\psi(K) \neq 0$, a result conjectured in [32].

Theorem 1.2. *If K is a transverse knot for which $sl(K) = s(K) - 1$ then $\psi(K) \neq 0$. The converse is also true if K belongs to a quasi-alternating (or any \widetilde{Kh} -thin¹) knot type. Here, $sl(K)$ is the self-linking number of the transverse knot, and $s(K)$ is Rasmussen's invariant [33].*

Corollary 1.3. *If K is a transverse representative of a quasi-alternating knot and $sl(K) = \sigma(K) - 1$ then $c(\xi_K) \neq 0$, and, hence, ξ_K is tight. Here, σ is the knot signature (with the convention that the right-handed trefoil has signature 2).*

Note that $sl(K) \leq s(K) - 1$ for any transverse knot K [31, 36]. Therefore, the hypothesis in Theorem 1.2 is equivalent to the sharpness of this upper bound for the self-linking number. In [22], Ng tabulates the maximal self-linking numbers for knots with at most 10 crossings. Combining those values with the results above, we can, in certain cases, establish the existence of a tight contact structure on the branched double cover of a given knot.

In Subsection 7.1, we provide several examples which demonstrate that our Khovanov-homological machinery is indeed useful and efficient for proving tightness. In particular, we show that ξ_K is tight but not Stein-fillable (our tightness result is therefore non-trivial) for several infinite families of transverse knots K which satisfy the hypotheses of Corollary 1.3. Some of the knots we consider are transverse 3-braids. When K is a transverse 3-braid, the contact structure ξ_K is compatible with a genus one, one boundary component open book, and the question of whether ξ_K is tight or overtwisted is resolved in [4, 13]. However, in contrast to the techniques used in [4, 13], our methods apply to transverse braids of arbitrary braid index, require no explicit calculation of the Heegaard Floer contact invariant, and are completely combinatorial.

Our construction may also be used to combinatorially prove that certain contact structures are not strongly symplectically fillable.

Proposition 1.4. *If $\widetilde{Kh}(S, \phi)$ is supported in non-positive homological gradings and $\psi(S, \phi) = 0$, then $c(S, \phi) = 0$, and, hence, $\xi_{S, \phi}$ is not strongly symplectically fillable [25].*

In contrast, if ϕ is composed solely of right-handed Dehn twists, then $\widetilde{Kh}(S, \phi)$ is supported in non-negative homological gradings and $\psi(S, \phi) \neq 0$. Recall that if L is a transverse link in S^3 , then the element $\psi(L) \in \widetilde{Kh}(L)$ is contained in the bi-grading $(0, sl(L) + 1)$ [32].

Corollary 1.5. *If L is a transverse link for which $\widetilde{Kh}(L)$ is supported in non-positive homological gradings and $\widetilde{Kh}(L)$ vanishes in the bi-grading $(0, sl(L) + 1)$, then ξ_L is not strongly symplectically fillable.*

In Subsection 7.3, we give examples which demonstrate the use of Proposition 1.4 and Corollary 1.5 in proving that certain contact structures are not strongly symplectically fillable.

We conclude with an investigation of some additional structure on the reduced Khovanov homology of an open book. Specifically, we show that $\psi(S, \phi)$ behaves naturally with respect to a comultiplication map on $\widetilde{Kh}(S, \phi)$. This closely parallels the behavior of the Ozsváth-Szabó contact invariant under a similar comultiplication defined on Heegaard Floer homology in [5].

¹A knot is said to be “ \widetilde{Kh} -thin” if its reduced Khovanov homology is supported in bi-gradings (i, j) , where $j - 2i$ is some fixed constant (see [21], for example).

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2. THE REDUCED KHOVANOV HOMOLOGY OF AN OPEN BOOK

Let S be a compact, oriented surface with boundary (we will use the notation $S_{k,r}$ when we wish to emphasize that S has genus k and r boundary components). Recall that the 3-manifold $M_{S,\phi}$ is defined to be $S \times [0, 1] / \sim$, where \sim is the identification given by

$$\begin{aligned} (x, 1) &\sim (\phi(x), 0), & x \in S \\ (x, t) &\sim (x, s), & x \in \partial S, \ t, s \in [0, 1]. \end{aligned}$$

Suppose that ϕ is a composition of Dehn twists, $\phi = D_{\gamma_1}^{\epsilon_1} \cdots D_{\gamma_n}^{\epsilon_n}$ with $\epsilon_j \in \{-1, 1\}$ (writing monodromy as composition $\phi = hg$, we assume that h is performed first). Choose points $0 = t_1 < \cdots < t_n = 1$ in the interval $[0, 1]$. Then $-M_{S,\phi}$ is obtained from $-M_{S,id}$ by performing ϵ_j -surgery on $\gamma_j \times \{t_j\}$, relative to the framing induced by S , for each $j = 1, \dots, n$. See Figure 2 for an example.

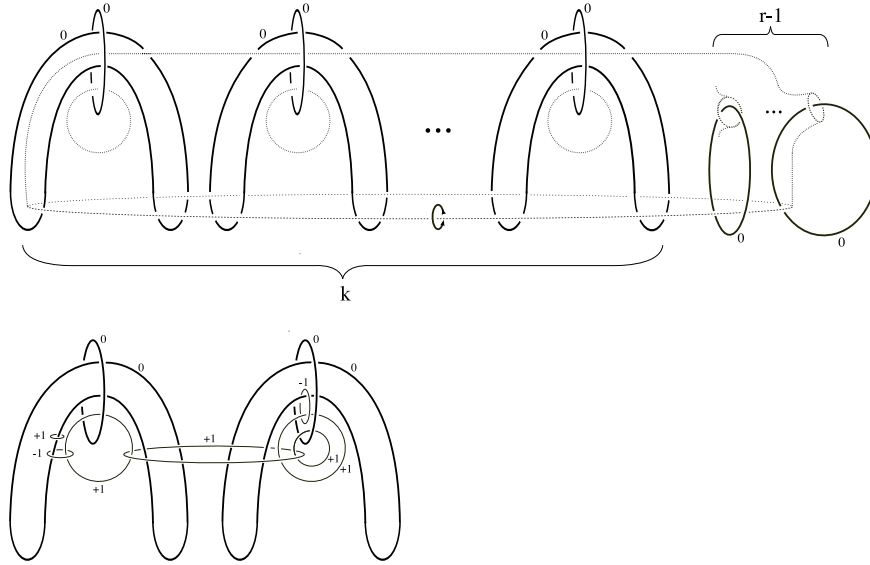


FIGURE 2. At the top is a surgery diagram for $-M_{S_{k,r},id}$. The dotted curves trace out a fiber $S_{k,r} \times \{t\}$. The boundary of this fiber is the binding B of the open book $(S_{k,r}, id)$. The arrow indicates the direction of increasing t values (that is, the S^1 direction of the fibration). At the bottom is a surgery diagram for $-M_{S_{2,1}, D_{\alpha_1} D_{\alpha_2} D_{\alpha_4} D_{\alpha_0}^{-1} D_{\alpha_4} D_{\alpha_1}^{-1} D_{\alpha_3}}$.

For each vector $i = (i_1, \dots, i_n) \in \{0, 1\}^n$ we define the “complete resolution” $(S, \phi)_i$ to be the 3-manifold obtained from $-M_{S,id}$ by performing $m(\epsilon_j, i_j)$ -surgery on $\gamma_j \times \{t_j\}$ for each $j = 1 \dots n$, where

$$m(\epsilon_j, i_j) = \begin{cases} 0, & \text{if } (\epsilon_j, i_j) = (1, 1) \text{ or } (-1, 0), \\ \infty, & \text{if } (\epsilon_j, i_j) = (1, 0) \text{ or } (-1, 1). \end{cases}$$

Said differently, in taking a complete resolution, we replace each ϵ_j -surgery in $-M_{S,\phi}$ with a 0- or ∞ -surgery as prescribed above (see Figure 3 for an example). This is analogous to

replacing each crossing in a planar link diagram with one of its two resolutions. We make this analogy more precise in Remark 2.3.

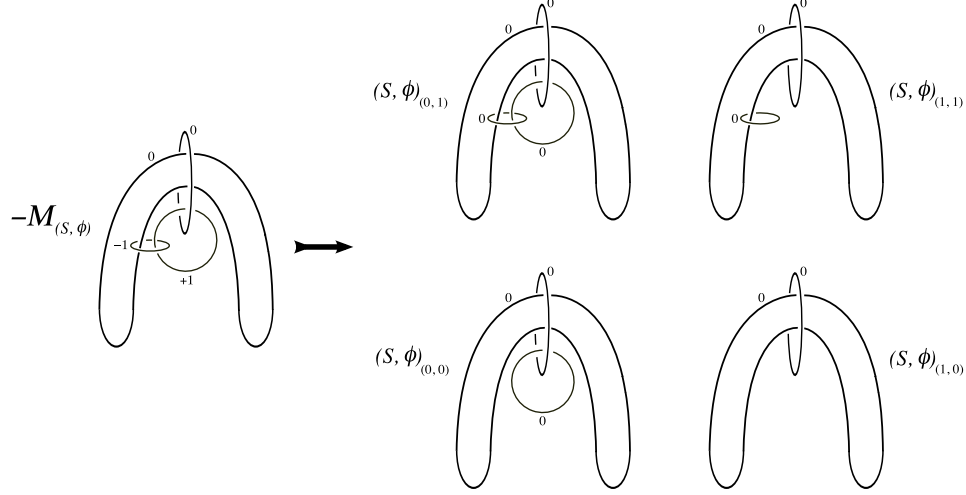


FIGURE 3. Surgery diagrams for the complete resolutions of the open book $(S, \phi) = (S_{1,1}, D_{\alpha_2} D_{\alpha_1}^{-1})$.

Following Ozsváth and Szabó, we construct a Heegaard multi-diagram compatible with all possible combinations of ϵ_j -, 0- and ∞ -surgeries on the components of the link $L = \bigcup_{j=1}^n \gamma_j \times \{t_j\}$, and we use this to build a chain complex $(C(S, \phi), D)$ whose differential counts holomorphic polygons in a symmetric product of this multi-diagram [30]. As a vector space,

$$C(S, \phi) = \bigoplus_{i \in \{0,1\}^n} \widehat{CF}((S, \phi)_i),$$

and D is the sum of maps

$$D_{i,i'} : \widehat{CF}((S, \phi)_i) \rightarrow \widehat{CF}((S, \phi)_{i'}),$$

over all pairs i, i' for which $i \leq i'$ (we say that $i \leq i'$ if $i_j \leq i'_j$ for all $j = 1, \dots, n$).

Theorem 2.1 ([30, Theorem 4.1]). *The homology $H_*(C(S, \phi), D)$ is isomorphic to $\widehat{HF}(-M_{S, \phi})$.*

There is a grading on $C(S, \phi)$ defined, for $x \in \widehat{CF}(-\Sigma(L_i))$, by $I(x) = |i| - n_-(\phi)$, where $|i| = i_1 + \dots + i_n$, and $n_-(\phi)$ is the number of left-handed Dehn twists in the composition ϕ . We refer to this as the “ I -grading” or the “homological grading” on $C(S, \phi)$. This grading induces an “ I -filtration” of the complex $(C(S, \phi), D)$, which, in turn, gives rise to a spectral sequence. Let $(E_I^k(S, \phi), D_I^k)$ denote the (E^k, D^k) term of this spectral sequence. The differential D_I^0 on the associated graded object $E_I^0(S, \phi)$ is the sum of the standard Heegaard Floer boundary maps

$$D_{i,i} : \widehat{CF}((S, \phi)_i) \rightarrow \widehat{CF}((S, \phi)_i).$$

Therefore, $E_I^1(S, \phi)$ is isomorphic to

$$\bigoplus_{i \in \{0,1\}^n} \widehat{HF}((S, \phi)_i).$$

The vector $i' \in \{0,1\}^n$ is said to be an “immediate successor” of i if $i'_k > i_k$ for some k and $i'_j = i_j$ for all $j \neq k$. If i' is an immediate successor of i , then $(S, \phi)_{i'}$ is obtained from $(S, \phi)_i$ by changing the surgery coefficient on the component $\gamma_k \times \{t_k\}$ from ∞ to 0 or from 0 to ∞ . In the first case,

$$(D_{i,i'})_* : \widehat{HF}((S, \phi)_i) \rightarrow \widehat{HF}((S, \phi)_{i'})$$

is the map induced by the 2-handle cobordism corresponding to 0-surgery on $\gamma_k \times \{t_k\}$. In the second case, $(D_{i,i'})_*$ is the map induced by the 2-handle cobordism corresponding to 0-surgery on a meridian of $\gamma_k \times \{t_k\}$. By construction, the differential D_I^1 on $\bigoplus_{i \in \{0,1\}^n} \widehat{HF}((S, \phi)_i)$ is the sum of the maps $(D_{i,i'})_*$, over all pairs i, i' for which i' is an immediate successor of i .

Definition 2.2. The *reduced Khovanov homology* of the open book (S, ϕ) is defined to be the graded vector space $E_I^2(S, \phi)$; it is denoted by $\widetilde{Kh}(S, \phi)$.

The reduced Khovanov homology of a link is a special case of our construction.

Remark 2.3. Suppose that $w = \sigma_{l_1}^{\epsilon_1} \cdots \sigma_{l_n}^{\epsilon_n}$ is a word in the generators of the braid group on m strands, with $\epsilon_j \in \{-1, 1\}$. For $i \in \{0,1\}^n$, let $(L_w)_i$ denote the link obtained from L_w by taking the i_j -resolution (see Figure 4) of the crossing corresponding to $\sigma_{l_j}^{\epsilon_j}$ for each $j = 1, \dots, n$. If $m = 2k + 1$, then $\Sigma(L_w)$ has an open book decomposition given by $(S_{k,1}, \phi_w)$, where $\phi_w = D_{\alpha_{l_1}}^{\epsilon_1} \cdots D_{\alpha_{l_n}}^{\epsilon_n}$. By design, the complete resolution $(S_{k,1}, \phi_w)_i$ is diffeomorphic to $-\Sigma((L_w)_i)$, and our entire construction of $\widetilde{Kh}(S_{k,1}, \phi_w)$ is identical to the construction of $\widetilde{Kh}(L_w)$ by Ozsváth and Szabó in [30]. Moreover, the homological grading on $\widetilde{Kh}(S_{k,1}, \phi_w)$ agrees with the homological grading on $\widetilde{Kh}(L_w)$.

The same is true for $m = 2k + 2$. The only difference in this case is that $\Sigma(L_w)$ has an open book decomposition given by $(S_{k,2}, \phi_w)$, where $\phi_w = D_{\alpha_{l_1}}^{\epsilon_1} \cdots D_{\alpha_{l_n}}^{\epsilon_n}$ is a composition of Dehn twists around the curves $\alpha_1, \dots, \alpha_{2k+1}$ depicted in Figure 5.

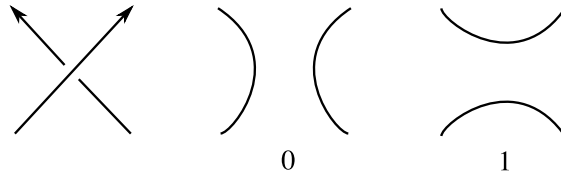


FIGURE 4. Above are the 0- and 1-resolutions of a positive crossing. For a negative crossing, the convention is reversed.

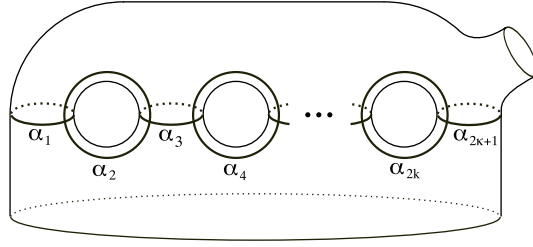


FIGURE 5. The curves $\alpha_1, \dots, \alpha_{2k+1}$ on the surface $S_{k,2}$.

3. INVARIANCE UNDER STABILIZATION

Recall that a positive (resp. negative) stabilization of the open book (S, ϕ) is an open book (S', ϕ') , where S' is the union of S with a 1-handle, and ϕ' is the composition of ϕ with a right-handed (resp. left-handed) Dehn twist around a curve in S' which intersects the co-core of the 1-handle exactly once.

Theorem 3.1. *If (S', ϕ') is a stabilization of (S, ϕ) , then $\widetilde{Kh}(S, \phi) \cong \widetilde{Kh}(S', \phi')$ as graded vector spaces.*

For the proof of Theorem 3.1, we need the following technical result from [30]. Recall that for any 3-manifold Y , $\widehat{HF}(Y)$ is a module over the algebra $\wedge^* H_1(Y)/\text{Tors}$ [28]. When $Y = \#^l(S^1 \times S^2)$, this module structure is particularly simple. In the following proposition, $Y_0(K)$ denotes the result of the 0-surgery on a knot $K \subset Y$.

Proposition 3.2 ([30, Proposition 6.1]). *If $Y = \#^l(S^1 \times S^2)$, then $\widehat{HF}(Y)$ can be identified with $\wedge^* H_1(Y)$ as an $\wedge^* H_1(Y)$ -module. If the knot K represents a circle in one of the $S^1 \times S^2$ summands of Y , then $Y' = Y_0(K)$ is diffeomorphic to $\#^{l-1}(S^1 \times S^2)$, and there is a natural identification*

$$\pi : H_1(Y)/[K] \rightarrow H_1(Y').$$

If W is the corresponding 2-handle cobordism from Y to Y' , then the induced map

$$F_W : \widehat{HF}(Y) \rightarrow \widehat{HF}(Y')$$

is specified by $F_W(\xi) = \pi(\xi)$. Dually, if $K \subset Y$ is an unknot, then $Y'' = Y_0(K)$ is diffeomorphic to $\#^{l+1}(S^1 \times S^2)$, and there is a natural inclusion

$$i : H_1(Y) \rightarrow H_1(Y'').$$

If W' is the corresponding 2-handle cobordism from Y to Y'' , then the induced map

$$F_{W'} : \widehat{HF}(Y) \rightarrow \widehat{HF}(Y'')$$

is specified by $F_{W'}(\xi) = i(\xi) \wedge [K'']$ where $[K''] \in H_1(Y'')$ generates the kernel of the map $H_1(Y'') \rightarrow H_1(W')$.

Our proof of Theorem 3.1 is similar in spirit to the proof that the reduced Khovanov homology of a link is invariant under the first Reidemeister move [17].

Proof of Theorem 3.1. Let $\phi = D_{\gamma_1}^{\epsilon_1} \cdots D_{\gamma_n}^{\epsilon_n}$. Suppose that (S', ϕ') is obtained from (S, ϕ) via stabilization, so that $\phi' = \phi \circ \tau_c$, where τ_c is a Dehn twist around a curve c which intersects the co-core of the new 1-handle exactly once. Let

$$L = \bigcup_{j=1}^n \gamma_j \times \{t_j\} \cup c \times \{t_{n+1}\}$$

be the framed link in $-M_{S', id}$ corresponding to the open book (S', ϕ') . Write

$$L = L' \cup c \times \{t_{n+1}\}.$$

There is a decomposition

$$-M_{S', id} \cong -M_{S, id} \# (S^1 \times S^2)$$

in which the sublink L' is contained in the $-M_{S,id}$ summand, while the component $c \times \{t_{n+1}\}$ represents a circle in the $S^1 \times S^2$ summand ($c \times \{t_{n+1}\}$ does not link L'). See Figure 6 for an example.

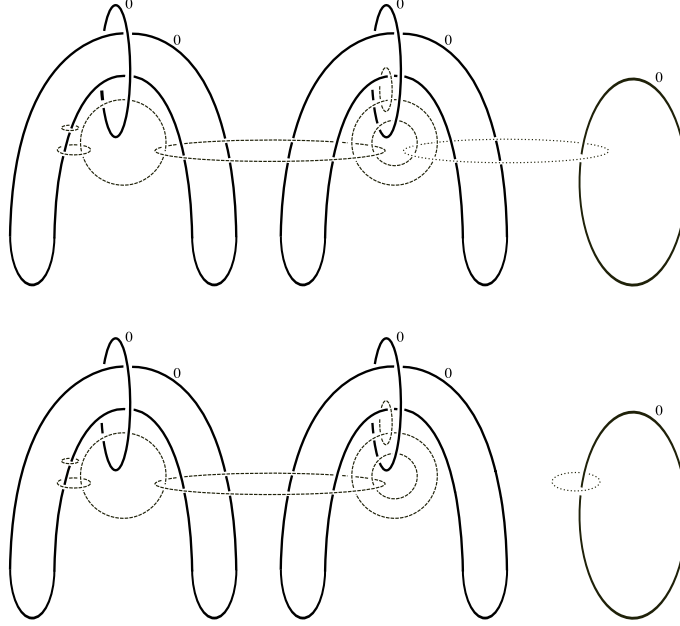


FIGURE 6. The open book in Figure 2 has a positive stabilization given by $(S_{2,2}, D_{\alpha_1} D_{\alpha_2} D_{\alpha_4} D_{\alpha_0}^{-1} D_{\alpha_4} D_{\alpha_1}^{-1} D_{\alpha_3} \circ D_c)$, where c is the curve $\alpha_5 \subset S_{2,2}$. The non-solid curves in this diagram represent the link L in $-M_{S_{2,2},id}$ corresponding to this stabilization. The component $c \times \{t_{n+1}\}$ is dotted and the sublink L' is dashed. The bottom figure is obtained from the top figure via handleslides, and represents the decomposition of $-M_{S_{2,2},id}$ into $-M_{S_{2,1},id} \# (S^1 \times S^2)$. When “sliding” a link component over a 0-surgery curve, we are simply taking the connected sum of the component with a (0-framed) longitude of the surgery curve. This longitude is an unknot, so the framed link is unchanged by the slide.

Suppose that (S', ϕ') is a negative stabilization of (S, ϕ) . Let $Y_0 \cong S^3$ and $Y_1 \cong S^1 \times S^2$ be the 3-manifolds obtained by performing 0- and ∞ -surgeries, respectively, on the component $c \times \{t_{n+1}\} \subset S^1 \times S^2$. Note that Y_1 is obtained from Y_0 by performing 0-surgery on a meridian m_c of $c \times \{t_{n+1}\}$. Let

$$C_- = \widehat{HF}(Y_0) \oplus \widehat{HF}(Y_1)$$

be the complex whose differential

$$d : \widehat{HF}(Y_0) \rightarrow \widehat{HF}(Y_1)$$

is the map induced by the 2-handle cobordism corresponding to 0-surgery on m_c .

Observe that $L' \subset -M_{S,id}$ is the link associated to the open book (S, ϕ) . For $i \in \{0, 1\}^{n+1}$, let \underline{i} be the vector in $\{0, 1\}^n$ defined by $\underline{i} = (i_1, \dots, i_n)$. Then,

$$(S', \phi')_i \cong (S, \phi)_{\underline{i}} \# Y_{i_n}$$

for all $i \in \{0, 1\}^{n+1}$, and the complex $E_I^1(S', \phi')$ decomposes as a tensor product,

$$E_I^1(S', \phi') \cong E_I^1(S, \phi) \otimes_{\mathbb{Z}_2} C_-.$$

Therefore,

$$(1) \quad \widetilde{Kh}(S', \phi') \cong \widetilde{Kh}(S, \phi) \otimes_{\mathbb{Z}_2} H_*(C_-, d)$$

as a vector space. Using the identification in Proposition 3.2,

$$C_- \cong \wedge^* H_1(S^3) \oplus \wedge^* H_1(S^1 \times S^2),$$

and the differential

$$d : \wedge^* H_1(S^3) \rightarrow \wedge^* H_1(S^1 \times S^2)$$

sends 1 to a generator of $\wedge^1 H_1(S^1 \times S^2)$, since m_c is an unknot in Y_0 . Therefore, $H_*(C_-, d)$ is generated by $\wedge^0 H_1(S^1 \times S^2) \cong \mathbb{Z}_2$. By Equation 1, $\widetilde{Kh}(S', \phi') \cong \widetilde{Kh}(S, \phi)$ as vector spaces, and it is clear that this isomorphism preserves the grading.

Now suppose that (S', ϕ') is a positive stabilization of (S, ϕ) . In this case, $Y_0 \cong S^1 \times S^1$ and $Y_1 \cong S^3$ are the 3-manifolds obtained via ∞ - and 0-surgeries on $c \times \{t_{n+1}\} \subset S^1 \times S^2$. Let

$$C_+ = \widehat{HF}(Y_0) \oplus \widehat{HF}(Y_1)$$

be the complex whose differential

$$d : \widehat{HF}(Y_0) \rightarrow \widehat{HF}(Y_1)$$

is the map induced by the 2-handle cobordism corresponding to 0-surgery on $c \times \{t_{n+1}\}$. As before,

$$(S', \phi')_i \cong (S, \phi)_{\underline{i}} \# Y_{i_n}$$

for all $i \in \{0, 1\}^{n+1}$, and the complex $E_I^1(S', \phi')$ decomposes as a tensor product,

$$E_I^1(S', \phi') \cong E_I^1(S, \phi) \otimes_{\mathbb{Z}_2} C_+.$$

Using the identification in Proposition 3.2,

$$C_+ \cong \wedge^* H_1(S^1 \times S^2) \oplus \wedge^* H_1(S^3),$$

and the differential

$$d : \wedge^* H_1(S^1 \times S^2) \rightarrow \wedge^* H_1(S^3)$$

sends 1 to 1, and kills $\wedge^1 H_1(S^1 \times S^2)$, since $c \times \{t_{n+1}\}$ is a circle in $Y_0 \cong S^1 \times S^2$. Therefore, $H_*(C_+, d)$ is generated by $\wedge^1 H_1(S^1 \times S^2) \cong \mathbb{Z}_2$, and $\widetilde{Kh}(S', \phi') \cong \widetilde{Kh}(S, \phi)$ as in the previous case. Let ξ be a generator of $\wedge^1 H_1(S^1 \times S^2)$. The chain map

$$\rho : E_I^1(S, \phi) \rightarrow E_I^1(S, \phi) \otimes_{\mathbb{Z}_2} C_+$$

defined by sending x to $x \otimes \xi$ induces this isomorphism since $\widetilde{Kh}(S', \phi')$ is generated by $E_I^1(S, \phi) \otimes \wedge^1 H_1(S^1 \times S^2)$.

□

4. THE ELEMENT $\psi(S, \phi)$ AND ITS RELATIONSHIP WITH $c(S, \phi)$.

Suppose that $\phi = D_{\gamma_1}^{\epsilon_1} \cdots D_{\gamma_n}^{\epsilon_n}$, and recall that

$$E_I^1(S, \phi) \cong \bigoplus_{i \in \{0,1\}^n} \widehat{HF}((S, \phi)_i).$$

Let i_o be the vector in $\{0,1\}^n$ defined by

$$(i_o)_j = \begin{cases} 0, & \text{if } \epsilon_j = 1, \\ 1, & \text{if } \epsilon_j = -1. \end{cases}$$

Then $(S, \phi)_{i_o}$ is the complete resolution obtained by performing ∞ -surgery on each curve $\gamma_j \times \{t_j\}$ in $-M_{S, id}$. If $S = S_{k,r}$ then $(S, \phi)_{i_o} \cong \#^{2k+r-1}(S^1 \times S^2)$. Therefore, we may identify $\widehat{HF}((S, \phi)_{i_o})$ with $\wedge^* H_1((S, \phi)_{i_o})$, by Proposition 3.2. Note that $\wedge^{2k+r-1} H_1((S, \phi)_{i_o}) \cong \mathbb{Z}_2$.

Definition 4.1. $\tilde{\psi}(S, \phi) \in E_I^1(S, \phi)$ is defined to be the generator of $\wedge^{2k+r-1} H_1((S, \phi)_{i_o})$.

It is not hard to show directly that $\tilde{\psi}(S, \phi)$ is closed in $(E_I^1(S, \phi), D_I^1)$. It is useful, however, to take a slightly more roundabout approach.

Lemma 4.2. *If (S', ϕ') is a positive stabilization of (S, ϕ) , then the chain map*

$$\rho : E_I^1(S, \phi) \rightarrow E_I^1(S', \phi')$$

defined in the proof of Theorem 3.1 sends $\tilde{\psi}(S, \phi)$ to $\tilde{\psi}(S', \phi')$. Recall that ρ induces an isomorphism from $\widetilde{Kh}(S, \phi)$ to $\widetilde{Kh}(S', \phi')$.

In particular, if (S', ϕ') is a positive stabilization of (S, ϕ) , then $\tilde{\psi}(S, \phi)$ is closed if and only if $\tilde{\psi}(S', \phi')$ is closed since the map ρ is injective.

Proof of Lemma 4.2. For $i \in \{0,1\}^n$, let \bar{i} be the vector in $\{0,1\}^{n+1}$ defined by $\bar{i} = (i_1, \dots, i_n, 0)$. Recall that

$$C_+ \cong \wedge^* H_1(S^1 \times S^2) \oplus \wedge^* H_1(S^3).$$

Restricted to the summand $\widehat{HF}((S, \phi)_{i_o})$,

$$\rho : \widehat{HF}((S, \phi)_{i_o}) \rightarrow \widehat{HF}((S, \phi)_{i_o}) \otimes_{\mathbb{Z}_2} C_+$$

sends $\tilde{\psi}(S, \phi)$ to $\tilde{\psi}(S, \phi) \otimes \xi$, where ξ is a generator of $\wedge^1 H_1(S^1 \times S^2)$. And, with respect to the identification

$$\widehat{HF}((S, \phi)_{i_o}) \otimes_{\mathbb{Z}_2} \wedge^* H_1(S^1 \times S^2) \cong \widehat{HF}((S', \phi')_{\bar{i}_o}),$$

$\tilde{\psi}(S, \phi) \otimes \xi$ corresponds to $\tilde{\psi}(S', \phi')$.

□

For the rest of this section, we will assume that S has connected boundary unless otherwise specified. Roberts' main observation in [35], transplanted to our more general setting, is that the binding B of the open book (S, ϕ) gives rise to an additional filtration of the complex $(C(S, \phi), D)$. More precisely, B may be incorporated into the Heegaard multi-diagram mentioned in Section 2 so that the intersection points in the multi-diagram which generate the summands $\widehat{CF}((S, \phi)_i)$ of $C(S, \phi)$ are each assigned an Alexander grading (as B gives rise

to a null-homologous knot B_i in each $(S, \phi)_i$. These Alexander gradings (or “ A -gradings” for short) induce a filtration of the group $C(S, \phi)$, which we call the “ A -filtration”. The differential D is a filtered map with respect to the A -filtration since none the curves $\gamma_j \times \{t_j\}$ algebraically link B [26, Section 8].

If C is a filtered group with filtration

$$\{0\} = \mathcal{F}_{j-l} \subset \dots \subset \mathcal{F}_{j-1} \subset \mathcal{F}_j = C,$$

then we refer to \mathcal{F}_k as “filtration level k ”. If d is a differential on C which respects this filtration, then there is an induced filtration of homology,

$$\{0\} = \mathcal{F}'_{j-l} \subset \dots \subset \mathcal{F}'_{j-1} \subset \mathcal{F}'_j = H_*(C, d),$$

where \mathcal{F}'_k is the set of homology classes which can be represented by cycles in \mathcal{F}_k .

The A -filtration of $(C(S, \phi), D)$ gives rise to an obvious filtration of $E_I^0(S, \phi)$, which, in turn, induces an A -filtration of each term $E_I^n(S, \phi)$.

Lemma 4.3. *$\tilde{\psi}(S, \phi)$ is the unique non-zero element of $E_I^1(S, \phi)$ in filtration level $-g(S)$. Moreover, $\tilde{\psi}(S, \phi)$ is closed in $(E_I^1(S, \phi), D_I^1)$.*

Every open book can be transformed into an open book with connected binding after a sequence of positive stabilizations. Therefore, Lemmas 4.2 and 4.3 imply that $\tilde{\psi}(S, \phi)$ is closed for any S .

Proof of Lemma 4.3. Let $S = S_{k,1}$. Then, $\tilde{\psi}(S, \phi)$ is the generator of $\wedge^{2k} H_1((S, \phi)_{i_o})$. The natural isomorphism between $\widehat{HF}((S, \phi)_{i_o})$ and $\widehat{HFK}((S, \phi)_{i_o}, B_{i_o})$ gives an identification of $\widehat{HFK}((S, \phi)_{i_o}, B_{i_o}, j)$ with the group $\wedge^{k-j} H_1((S, \phi)_{i_o})$ [26]. As a result, $\tilde{\psi}(S, \phi)$ corresponds to the generator of $\widehat{HFK}((S, \phi)_{i_o}, B_{i_o}, -k)$, and is therefore a non-zero element of $E_I^1(S, \phi)$ in filtration level $-k = -g(S)$.

If $i \neq i_o$ then $(S, \phi)_i$ is obtained from $M_{S,id}$ by performing 0-surgery on at least one of the curves $\gamma_j \times \{t_j\}$. Therefore, $g(B_i) < g(S)$, and, hence, $\widehat{HFK}((S, \phi)_i, B_i, j)$ vanishes for $j \leq -g(S)$. As a result, there is no non-zero element of $\widehat{HF}((S, \phi)_i)$ in A -filtration level $-g(S)$. Since the differential D_I^1 is a filtered map with respect to the A -filtration of $E_I^1(S, \phi)$, it follows that $D_I^1(\tilde{\psi}(S, \phi)) = 0$.

□

Definition 4.4. For any open book (S, ϕ) , we define $\psi(S, \phi)$ to be the image of $\tilde{\psi}(S, \phi)$ in $E_I^2(S, \phi) = \widetilde{Kh}(S, \phi)$.

Note that $\psi(S, \phi)$ is contained in homological grading 0. Below, we show that ψ vanishes for negative stabilizations.

Lemma 4.5. *Let (S, ϕ) be any open book, and suppose that (S', ϕ') is a negative stabilization of (S, ϕ) . Then $\psi(S', \phi') = 0$.*

Proof of Lemma 4.5. As in the proof of Theorem 3.1,

$$E_I^1(S', \phi') \cong E_I^1(S, \phi) \otimes_{\mathbb{Z}_2} C_-,$$

where C_- is the complex

$$C_- \cong \wedge^* H_1(S^3) \oplus \wedge^* H_1(S^1 \times S^2),$$

whose differential

$$d : \wedge^* H_1(S^3) \rightarrow \wedge^* H_1(S^1 \times S^2)$$

sends 1 to a generator ξ of $\wedge^1 H_1(S^1 \times S^2)$. Under this identification, the summand $\widehat{HF}((S', \phi')_{i_o})$ is identified with

$$\widehat{HF}((S, \phi)_{i_o}) \otimes_{\mathbb{Z}_2} \wedge^* H_1(S^1 \times S^2),$$

and $\tilde{\psi}(S', \phi')$ corresponds to $\tilde{\psi}(S, \phi) \otimes \xi$. Since $\tilde{\psi}(S, \phi)$ is closed in $E_I^1(S, \phi)$, $\tilde{\psi}(S, \phi) \otimes \xi$ is the boundary of $\tilde{\psi}(S, \phi) \otimes 1$ in this complex. Therefore, $\psi(S', \phi') = 0$. \square

Remark 4.6. By Theorem 3.1 and Lemma 4.2, it is enough to prove Theorem 1.1 for open books with connected binding.

Remark 4.7. It is not hard to see directly that the invariant of the transverse $(2k+1)$ -braid L_w defined by the second author in [32] is precisely the class $\psi(S_{k,1}, \phi_w)$ in $\widehat{Kh}(S_k, \phi_w) \cong \widehat{Kh}(L_w)$. This also follows from Lemma 4.3 combined with Roberts' characterization of $\psi(L_w)$ in [35].

Let us denote by $E_A^n(S, \phi)$ the E^n term of the spectral sequence associated to the A -filtration of $(C(S, \phi), D)$. In [35], Roberts (effectively) proves that, for $n \geq 1$, $E_A^n(S, \phi)$ is isomorphic as a graded group to the E^n term of the spectral sequence associated to the Alexander filtration of $\widehat{CF}(-M_{S,\phi})$ induced by B [35, Lemma 7]. We may therefore view the contact invariant $c(S, \phi)$ as the unique generator of $E_A^\infty(S, \phi)$ in A -grading $-g(S)$ [29]. Equivalently, $c(S, \phi)$ is the generator of the $A = -g(S)$ filtration level of $H_*(C(S, \phi), D)$.

In order to make the relationship between $\psi(S, \phi)$ and $c(S, \phi)$ more transparent, we provide a short review of the “cancellation lemma,” and describe how it is used to compute spectral sequences.

Lemma 4.8 (see [34, Lemma 5.1]). *Suppose that (C, d) is a complex over \mathbb{Z}_2 , freely generated by elements x_i , and let $d(x_i, x_j)$ be the coefficient of x_j in $d(x_i)$. If $d(x_k, x_l) = 1$, then the complex (C', d') with generators $\{x_i | i \neq k, l\}$ and differential*

$$d'(x_i) = d(x_i) + d(x_i, x_l)d(x_k)$$

is chain homotopy equivalent to (C, d) . The chain homotopy equivalence is induced by the projection $\pi : C \rightarrow C'$, while the equivalence $\iota : C' \rightarrow C$ is given by $\iota(x_i) = x_i + d(x_i, x_l)x_k$.

We say that (C', d') is obtained from (C, d) by “canceling” the component of the differential d from x_k to x_l . Lemma 4.8 admits a refinement for filtered complexes. In particular, suppose that there is a grading on C which induces a filtration of the complex (C, d) , and let the elements x_i be homogeneous generators of C . If $d(x_k, x_l) = 1$, and x_k and x_l have the same grading, then the complex obtained by canceling the component of d from x_k to x_l is *filtered* chain homotopy equivalent to (C, d) since both π and ι are filtered maps in this case.

Computing the spectral sequence associated to such a filtration is the process of performing cancellation in a series of stages until we arrive at a complex in which the differential is zero (the E^∞ term). The E^n term records the result of this cancellation after the n th stage. Specifically,

the E^0 term is simply the graded vector space $C = \bigoplus C_i$. The E^1 term is the graded vector space $C^{(1)}$, where $(C^{(1)}, d^{(1)})$ is obtained from (C, d) by canceling the components of d which do not shift the grading. For $n > 1$, the E^n term is the graded vector space $C^{(n)}$, where $(C^{(n)}, d^{(n)})$ is obtained from $(C^{(n-1)}, d^{(n-1)})$ by canceling the components of $d^{(n-1)}$ which shift the grading by $n - 1$. See Figure 7 for an illustration of this process (in this diagram, the generators are represented by dots and the components of the differential are represented by arrows).

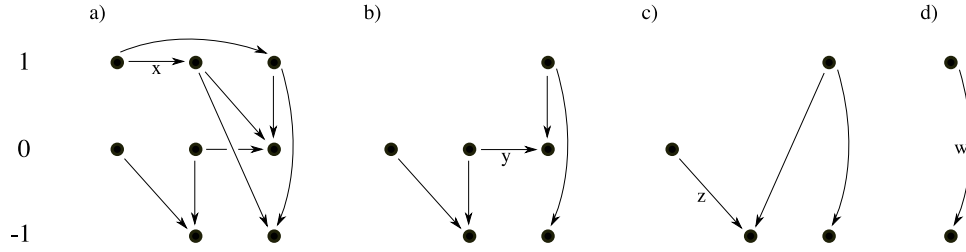


FIGURE 7. The diagram in a) represents a graded complex C , where the grading of a generator is given by 1, 0, or -1 . This grading induces a filtration $\mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 = C$. The complex in b) is obtained from that in a) by canceling the component x of the differential. The complex in c) is obtained from that in b) by canceling y . This graded vector space represents the E^1 term of the spectral sequence associated to the filtration of C . The complex in d) is obtained from that in c) by canceling z , and it represents the E^2 term of the spectral sequence. The $E^3 = E^\infty$ term of the spectral sequence is trivial, and is obtained from the complex in d) by canceling w .

Let us now apply the cancellation lemma to the complex $(C(S, \phi), D)$. As mentioned above, $(C(S, \phi), D)$ is generated by the intersection points of the Heegaard multi-diagram from Section 2. These generators are homogeneous with respect to both the I -grading and the A -grading on $C(S, \phi)$. Canceling all components of D between generators in the same (I, A) -bi-grading, we obtain a complex $(C(S, \phi)', D')$ which is bi-filtered chain homotopy equivalent to $(C(S, \phi), D)$ (since π and ι are both bi-filtered maps). Let us denote by $E_I^n(S, \phi)'$ and $E_A^n(S, \phi)'$ the E^n terms of the spectral sequences associated to the I - and A -filtrations of $(C(S, \phi)', D')$ (these terms are isomorphic as graded groups to $E_I^n(S, \phi)$ and $E_A^n(S, \phi)$).

As a bi-filtered group, $C(S, \phi)'$ is clearly isomorphic to

$$\bigoplus_{i \in \{0,1\}^n} \widehat{HFK}((S, \phi)_i, B_i).$$

Consequently, the generator c of $\widehat{HFK}((S, \phi)_{i_o}, B_{i_o}, -g(S))$ is the unique non-zero element of $C(S, \phi)'$ in A -filtration level $-g(S)$ (see the proof of Lemma 4.3). Thus, according to Definition 4.4, the element $\psi(S, \phi)$ may be viewed as the image of c in $E_I^2(S, \phi)' \cong \widehat{Kh}(S, \phi)$. Similarly, the contact invariant $c(S, \phi)$ may be viewed as the image of c in $H_*(C(S, \phi)', D') \cong \widehat{HF}(-M_{S, \phi})$. This is the true sense in which $\psi(S, \phi)$ “corresponds” to $c(S, \phi)$.

In this context, Theorem 1.1 boils down to the statement that if the spectral sequence associated to the I -filtration of $(C(S, \phi)', D')$ collapses at $E_I^2(S, \phi)'$ and the image of c in $E_I^2(S, \phi)'$ is non-zero, then the image of c in $H_*(C(S, \phi)', D')$ is non-zero.

Proof of Theorem 1.1. Let us assume that the spectral sequence associated to the I -filtration of $(C(S, \phi)', D')$ collapses at $E_I^2(S, \phi)'$, and suppose that c is exact in $(C(S, \phi)', D')$. Cancel all components of D' between generators in the same I -grading to obtain a new complex $(C(S, \phi)'', D'')$, where $C(S, \phi)'' = E_I^1(S, \phi)'$. Note that there is no component of D' between two generators of $\widehat{HFK}((S, \phi)_{i_o}, B_{i_o})$ since this group is isomorphic to $\widehat{HF}((S, \phi)_{i_o})$. Therefore, c represents an exact element of $(C(S, \phi)'', D'')$.

Now, cancel all components of D'' from generators x_k to x_l , where $x_l \neq c$ and $I(x_l) = I(x_k) + 1$. The element c remains exact in the resulting complex $(C(S, \phi)''', D''')$, and any component of D''' which shifts the I -grading by 1 must map to c . If such a component exists, then the image of c is zero in the group $E_I^2(S, \phi)'$ (which is obtained from $C(S, \phi)'''$ by canceling this component), and we are done. If no such component exists, then $C(S, \phi)''' = E_I^2(S, \phi)'$, which implies that D''' is trivial since we are assuming that the spectral sequence collapses at $E_I^2(S, \phi)'$. But this contradicts the fact that c is exact in $(C(S, \phi)''', D''')$. \square

Remark 4.9. One should not read too much into the “correspondence” between $\psi(S, \phi)$ and $c(S, \phi)$. For instance, it is possible that $\psi(S, \phi) = 0$ while $c(S, \phi) \neq 0$, perhaps even when the spectral sequence associated to the I -filtration of $(C(S, \phi)', D')$ collapses at $E_I^2(S, \phi)'$. An example of this sort of phenomenon is given by the model complex (C, d) generated by the three elements $x_{0,0}, y_{1,-2}, z_{2,-1}$, with $dx = y + z$ (here, the subscripts indicate the (I, A) -bi-grading). The element y in the lowest A -grading plays the role of c . The E^1 term of the spectral sequence associated to the I -filtration of (C, d) is generated by the elements x, y, z as well, and $d^1(x) = y$ (think $\psi(S, \phi) = 0$). Yet, y is not a boundary in (C, d) (think $c(S, \phi) \neq 0$). Moreover, this spectral sequence collapses at the E^2 term.

The hypotheses in Proposition 1.4 are designed to avoid the type of phenomenon described in Remark 4.9. Applied to the complex $(C(S, \phi)', D')$, Proposition 1.4 says that if the image of c in $E_I^2(S, \phi)'$ is zero and $E_I^2(S, \phi)'$ is supported in non-positive I -gradings, then the image of c in $H_*(C(S, \phi)', D')$ is zero.

Proof of Proposition 1.4. Suppose that $\phi = D_{\gamma_1}^{\epsilon_1} \cdots D_{\gamma_n}^{\epsilon_n}$, and let $K = n - n_-(\phi)$. If \mathcal{F}_j is the subset of $C(S, \phi)'$ generated by homogeneous elements with I -gradings $\geq j$, then

$$\{0\} = \mathcal{F}_{K+1} \subset \mathcal{F}_K \subset \cdots \subset \mathcal{F}_{-n_-(\phi)} = C(S, \phi)'.$$

Let us assume that $E_I^2(S, \phi)'$ is supported in non-positive I -gradings. If the image of c in $E_I^2(S, \phi)'$ is zero, then there must exist some y with $I(y) = -1$ such that $D'(y) = c + x$, where $x \in \mathcal{F}_1$. Let k be the greatest integer for which there exists some y' such that $D'(y') = c + x'$, where $x' \in \mathcal{F}_k$. We will show that $k = K + 1$, which implies that $x' = 0$, and, hence, that c is a boundary in $(C(S, \phi)', D')$.

Suppose, for a contradiction, that $k < K + 1$. Write $x' = x_k + x''$, where $I(x_k) = k$, and $x'' \in \mathcal{F}_{k+1}$. Note that $D'(x_k + x'') = 0$ as $x' = x_k + x''$ is homologous to c . Since every component of D' shifts the I -grading by at least 1, it follows that $D'(x'') \in \mathcal{F}_{k+2}$. But this

implies that $D'(x_k) \in \mathcal{F}_{k+2}$ as well, since $D'(x_k + x'') = 0$. Therefore, x_k represents a cycle in $(E_I^1(S, \phi)', (D'_I)^1)$. Since $k \geq 1$ and $E_I^2(S, \phi)'$ is supported in non-positive I -gradings, it must be that x_k is also a boundary in $(E_I^1(S, \phi)', (D'_I)^1)$. That is, there is some y'' with $I(y'') = k - 1$ such that $D'(y'') = x_k + x'''$, where $x''' \in \mathcal{F}_{k+1}$. But then, $D'(y' + y'') = c + (x'' + x''')$, and the fact that $x'' + x'''$ is contained in \mathcal{F}_{k+1} contradicts our earlier assumption on the maximality of k .

□

5. THE COMPUTABILITY OF $\widetilde{Kh}(S, \phi)$ AND $\psi(S, \phi)$

As was mentioned in the introduction, the mapping class group of $S_{k,1}$ is generated by Dehn twists around the curves $\alpha_0, \dots, \alpha_{2k}$ depicted in Figure 1. In this section, we show that $\widetilde{Kh}(S_{k,1}, \phi)$ and $\psi(S_{k,1}, \phi)$ are combinatorially computable when ϕ is a composition of Dehn twists around the curves shown in Figure 8, which include $\alpha_0, \dots, \alpha_{2k}$.

According to Proposition 3.2, if $Y = \#^l(S^1 \times S^2)$, then the map on Heegaard Floer homology induced by a 2-handle cobordism corresponding to 0-surgery on an unknot in Y or on a circle in an $S^1 \times S^2$ summand of Y depends only on homological data. The combinatoriality of $\widetilde{Kh}(S_{k,1}, \phi)$ and $\psi(S_{k,1}, \phi)$ therefore follows directly from the lemma below.

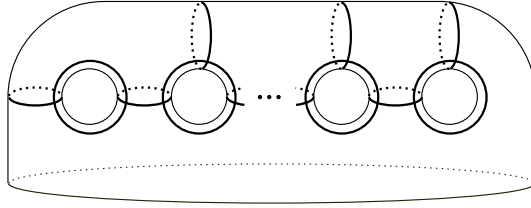


FIGURE 8.

Lemma 5.1. *Suppose that ϕ is a composition of Dehn twists around the curves depicted in Figure 8. Then, each $(S_{k,1}, \phi)_i$ is diffeomorphic to $\#^l(S^1 \times S^2)$ for some l . Moreover, if i' is an immediate successor of i , then the map*

$$(D_{i,i'})_* : \widehat{HF}((S_{k,1}, \phi)_i) \rightarrow \widehat{HF}((S_{k,1}, \phi)_{i'})$$

is induced by a 2-handle cobordism corresponding to 0-surgery on either an unknot in $(S_{k,1}, \phi)_i$ or on a circle in one of the $S^1 \times S^2$ summands of $(S_{k,1}, \phi)_i$.

Proof of Lemma 5.1. Let ϕ be the composition $\phi = D_{\gamma_1}^{\epsilon_1} \cdots D_{\gamma_n}^{\epsilon_n}$, where each γ_j is among the curves depicted in Figure 8. As described in Section 2, a surgery diagram for the complete resolution $(S_{k,1}, \phi)_i$ is obtained from the diagram for $-M_{S_{k,1}, id}$ (depicted in Figure 2) by performing 0-surgeries on the curves in some subset of $\{\gamma_j \times \{t_j\}\}_{j=1}^n$.

After isotopy, such a diagram consists of several “blocks” of concentric circles, together with “staggered” horizontal and vertical circles (see Figure 9). Consider the slightly more general arrangement of curves represented schematically in Figure 10. In this schematic picture, the shaded annuli represent blocks of concentric circles, the unmarked rectangles represent staggered horizontal and vertical circles, and the rectangle labeled A represents a union of horizontal circles which may be slid past one another like beads on an abacus, up and down the strands of the concentric circles in the rightmost block. Certainly, the curves in the surgery diagram for $(S_{k,1}, \phi)_i$ constitute such an arrangement.

Our proof that $(S_{k,1}, \phi)_i$ is a connected sum of $S^1 \times S^2$'s is rather inelegant. We start with a diagram obtained by performing 0-surgery on each of the curves in an arrangement as in Figure 10, and perform handleslides and handle cancellations until we arrive at a diagram for 0-surgeries on the components of an unlink.

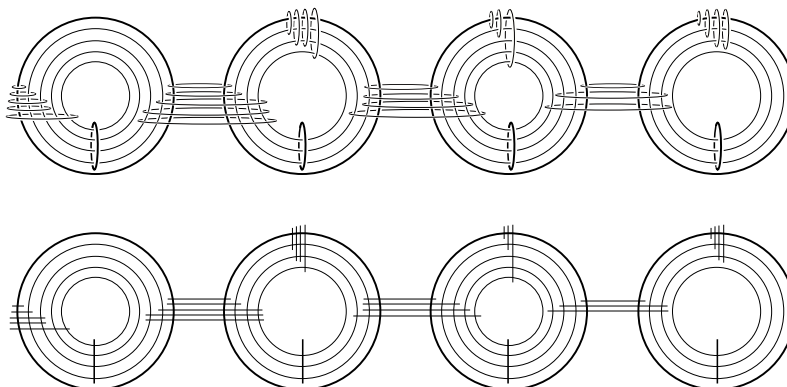


FIGURE 9. At the top is an example of the surgery diagram for a complete resolution $(S_{4,1}, \phi)_i$. Though we have not labeled the surgery coefficients, it is understood that they are all 0. The thicker circles are the surgery curves for $-M_{S_{4,1}, id}$. The bottom picture is shorthand for the surgery diagram at the top.

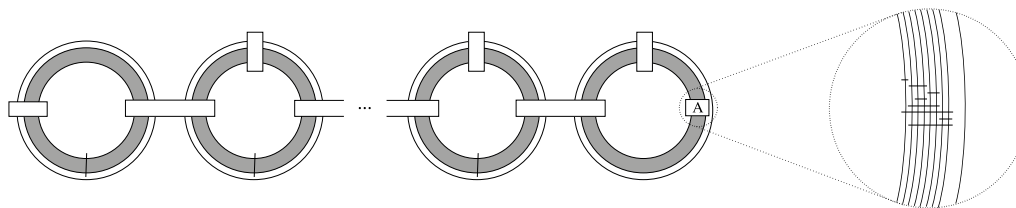


FIGURE 10. The magnified region is an example of what the horizontal circles represented by the rectangle A might look like. The key point is that you can slide any one of them past any other.

We perform these handleslides/cancellations beginning with the rightmost block of concentric circles. If C is the innermost of these concentric circles, then C might link some of the horizontal and vertical circles represented by the rectangles which intersect the rightmost annulus in Figure 10. If there is no such linking, then C is an unknot which may be pulled aside. If there *is* linking, then handleslide these horizontal and vertical circles over one another until only one remains which links C (it does not matter which circle is handleslid over which), and then cancel C with this remaining surgery curve (see Figure 11 for an illustration of this process). In either case, we are left with one fewer concentric circle in our arrangement. It is not hard to see that this procedure may be repeated until all of the concentric circles in the rightmost block have been cancelled or pulled aside from the arrangement. Afterwards, what remains is an arrangement of surgery curves as in Figure 10 with one fewer block of concentric circles, together with an unlink. Repeat this process until all of the blocks have been eliminated, and all that is left is an unlink.

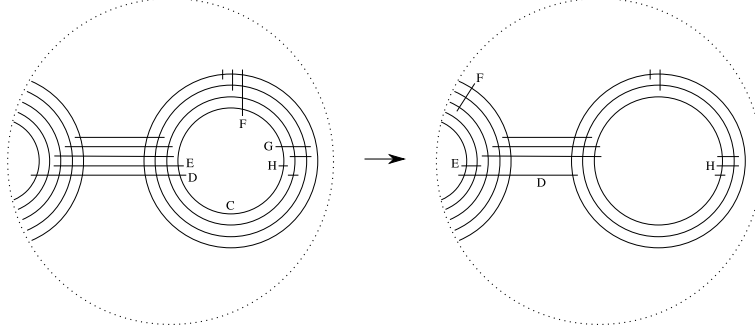


FIGURE 11. In order, handleslide F over E , then E over D , then D over G , and, finally, H over D . Next, cancel C with G . The result is the picture on the right. We can iterate this process until we have eliminated all of the concentric circles in the rightmost block.

As was described in Section 2, the maps

$$(D_{i,i'})_* : \widehat{HF}((S, \phi)_i) \rightarrow \widehat{HF}((S, \phi)_{i'})$$

are induced by 2-handle cobordisms corresponding either to 0-surgery on $\gamma_k \times \{t_k\}$ for some k , or on a meridian of $\gamma_k \times \{t_k\}$. Therefore, to finish the proof of Lemma 5.1, it suffices to show that if K is one of the curves in the arrangement in Figure 10, then K , thought of as a knot in the 3-manifold Y obtained by performing 0-surgery on all of the *other* curves in the arrangement, is either an unknot (with the correct framing) or a circle in one of the $S^1 \times S^2$ summands of Y . But this can be seen explicitly by keeping track of the knot K as one performs the reductive algorithm described above. \square

In order to actually compute $\widetilde{Kh}(S_{k,1}, \phi)$, where ϕ is a composition of Dehn twists around the curves in Figure 8, begin by fixing a basis for the first homology of each complete resolution. By Proposition 3.2 and Lemma 5.1,

$$\widehat{HF}((S_{k,1}, \phi)_i) \cong \wedge^* H_1((S_{k,1}, \phi)_i).$$

Now suppose that i' is an immediate successor of i for which $i'_k > i_k$, and determine how the basis for $H_1((S_{k,1}, \phi)_i)$ changes upon performing 0-surgery on either $\gamma_k \times \{t_k\}$ or its meridian. With this information in hand, we may use (according to Lemma 5.1) Proposition 3.2 to compute the map

$$(D_{i,i'})_* : \wedge^* H_1((S, \phi)_i) \rightarrow \wedge^* H_1((S, \phi)_{i'})$$

in terms of the bases that we fixed at the beginning. This is how the program `Kh` [3] works. Below, we give a simple example of this procedure.

Example 5.2. On the left of Figure 12 are surgery diagrams for the complete resolutions of the open book $(S, \phi) = (S_{1,1}, D_{\alpha_2} D_{\alpha_1}^{-1})$. We have omitted the surgery coefficients, but it is understood that they are all 0 (compare with Figure 3). Instead, we have labeled the surgery curves A , B , C , and D . Let a , b , c , and d denote the respective meridians of these

curves. The right side of Figure 12 depicts the term $E_I^1(S, \phi) \cong \bigoplus_{i \in \{0,1\}^2} \widehat{HF}((S, \phi)_i)$ using the identification of $\widehat{HF}((S, \phi)_i)$ with $\wedge^* H_1((S, \phi)_i)$ described in Proposition 3.2. In addition, we have indicated the maps $(D_{i,i'})_*$ which comprise the differential D_I^1 .

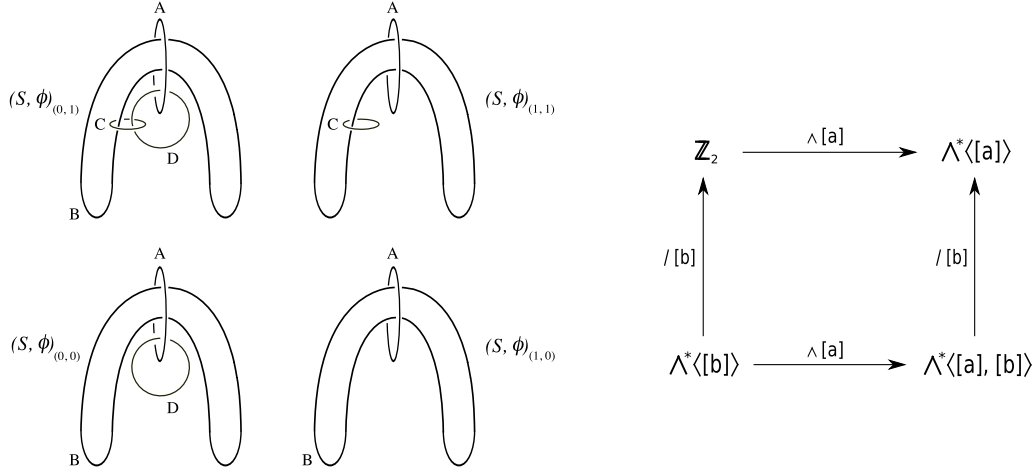


FIGURE 12. Computing the reduced Khovanov homology of $(S, \phi) = (S_{1,1}, D_{\alpha_2} D_{\alpha_1}^{-1})$.

Observe that $(S, \phi)_{(0,0)} \cong S^1 \times S^2$, and its first homology is generated by $[b]$. Therefore, $\widehat{HF}((S, \phi)_{(0,0)})$ may be identified $\wedge^*\langle [b] \rangle$. Meanwhile, $(S, \phi)_{(1,0)} \cong \#^2(S^1 \times S^2)$, and its first homology is generated by $[a]$ and $[b]$; hence, $\widehat{HF}((S, \phi)_{(1,0)})$ may be identified with $\wedge^*\langle [a], [b] \rangle$. Now, $(S, \phi)_{(1,0)}$ is obtained from $(S, \phi)_{(0,0)}$ by performing 0-surgery on the meridian d . Handlesliding d over A , we see that d is an unknot in $(S, \phi)_{(0,0)}$. If W is the 2-handle cobordism corresponding to this 0-surgery, then the kernel of the map $H_1((S, \phi)_{(1,0)}) \rightarrow H_1(W)$ is generated by $[a]$. Therefore, by Proposition 3.2, the map

$$(D_{(0,0),(1,0)})_* : \wedge^*\langle [b] \rangle \rightarrow \wedge^*\langle [a], [b] \rangle$$

sends x to $x \wedge [a]$. The other components $(D_{i,i'})_*$ are computed similarly.

It is easy to see that $\widehat{Kh}(S, \phi)$ is isomorphic to \mathbb{Z}_2 , and is generated by the cycle corresponding to $[b] \in \widehat{HF}((S, \phi)_{(1,0)})$. Moreover, the element $\tilde{\psi}(S, \phi) = [a] \wedge [b] \in \widehat{HF}((S, \phi)_{(1,0)})$ is the boundary of $[b] \in \widehat{HF}((S, \phi)_{(0,0)})$ under the differential D_I^1 . Therefore, $\psi(S, \phi) = 0$. These results are not surprising since $\widehat{Kh}(S, \phi)$ is simply the reduced Khovanov homology of the closed braid $L_{\sigma_2 \sigma_1^{-1}}$, which is the unknot. In addition, $\psi(S, \phi)$ is Plamenevskaya's transverse link invariant, $\psi(L_{\sigma_2 \sigma_1^{-1}})$, which is known to vanish [32, Proposition 3]. (Of course, vanishing of $\psi(S, \phi) = 0$ also follows from Lemma 4.5, since the open book (S, ϕ) can be obtained by negative stabilization.)

Our generalization of reduced Khovanov homology was motivated, in part, by a conjecture of Ozsváth which suggests that $\text{rk}(\widehat{Kh}(K_1)) = \text{rk}(\widehat{Kh}(K_2))$ whenever $\Sigma(K_1) \cong \Sigma(K_2)$.² This

²Liam Watson has recently found an infinite family of counterexamples [39].

conjecture would follow from our construction if $\widetilde{Kh}(S, \phi)$ were an invariant of the 3-manifold $M_{S, \phi}$. Although $\widetilde{Kh}(S, \phi)$ is invariant under stabilization, $\widetilde{Kh}(S, \phi)$ is not an invariant of the isotopy class of ϕ . This can be shown using the program **Kh**.

According to [15] (see also [24]), the mapping class group of $S_{k,1}$ has a presentation

$$\langle D_{\alpha_0}, \dots, D_{\alpha_{2k}} | b_0, c_1, \dots, c_{2k-1}, r_1, r_2 \rangle,$$

where b_0 is the relation

$$(D_{\alpha_4} D_{\alpha_0} D_{\alpha_4})(D_{\alpha_0} D_{\alpha_4} D_{\alpha_0})^{-1},$$

c_j is the relation

$$(D_{\alpha_j} D_{\alpha_{j+1}} D_{\alpha_j})(D_{\alpha_{j+1}} D_{\alpha_j} D_{\alpha_{j+1}})^{-1},$$

and r_1 is the relation $\phi_1 \phi_2^{-1}$, where

$$\phi_1 = (D_{\alpha_1} D_{\alpha_2} D_{\alpha_3})^4 D_{\alpha_4}^{-1} D_{\alpha_3}^{-1} D_{\alpha_2}^{-1},$$

and

$$\phi_2 = D_{\alpha_0} D_{\alpha_4}^{-1} D_{\alpha_3}^{-1} D_{\alpha_2}^{-1} D_{\alpha_1}^{-1} D_{\alpha_0} D_{\alpha_1}^{-1} D_{\alpha_4} D_{\alpha_2}^{-1} D_{\alpha_3} D_{\alpha_2} D_{\alpha_4}^{-1} D_{\alpha_1} D_{\alpha_0}^{-1} D_{\alpha_1}.$$

The relation r_2 is more complicated. Computer calculations suggest that $\widetilde{Kh}(S_{k,1}, \phi)$ is invariant under composition with the relations b_0 and c_j , but not with r_1 , since $\widetilde{Kh}(S_{2,1}, \phi_1) \cong \mathbb{Z}_2^6$, while $\widetilde{Kh}(S_{2,1}, \phi_2) \cong \mathbb{Z}_2^{25}$.

For a while, we had hoped that the *vanishing* of $\psi(S, \phi)$ was an invariant the contact structure $\xi_{S, \phi}$. We now know this to be false. Assume, for a contradiction, that it is true. Then $\psi(S, \phi) = 0$ whenever $\xi_{S, \phi}$ is overtwisted. For, if $\xi_{S, \phi}$ is overtwisted, then $\xi_{S, \phi}$ is compatible with an open book (S, ϕ') which is a negative stabilization of some other open book (see [24]). By Lemma 4.5, $\psi(S, \phi') = 0$. On the other hand, the open book $(S, \phi) = (S_{2,1}, D_{\alpha_1} D_{\alpha_2} D_{\alpha_3} D_{\alpha_5} (D_{\alpha_3})^{-5} D_{\alpha_4} D_{\alpha_0})$ in Example 7.9 below corresponds to an overtwisted contact structure, while the program **Trans** [3] shows that $\psi(S, \phi) \neq 0$.

6. TRANSVERSE LINKS

In this section, we prove Theorem 1.2 and Corollary 1.3. In Section 7, we will use these two results in conjunction with Theorem 1.1 to show that the contact structure ξ_K is tight for several classes of transverse knots $K \subset S^3$.

Proof of Theorem 1.2. Suppose that K is a transverse knot with $sl(K) = s(K) - 1$. Though we are interested in reduced Khovanov homology with coefficients in \mathbb{Z}_2 , it is instructive to first consider the case of non-reduced homology with rational coefficients. In [18], Lee introduces a differential $d' = d + \Phi$ on the Khovanov chain complex $CKh(K)$, where d is Khovanov's original differential [17], and Φ is a map which raises the quantum grading (or "q-grading" for short). Recall that d' arises from the multiplication and comultiplication maps defined by

$$\begin{aligned} m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= m(\mathbf{v}_- \otimes \mathbf{v}_-) = \mathbf{v}_+ & \Delta(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ \\ m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_- & \Delta(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- + \mathbf{v}_+ \otimes \mathbf{v}_+ \end{aligned}$$

(we follow the notation of [33], in which \mathbf{v}_- and \mathbf{v}_+ are the standard generators in quantum degrees -1 and $+1$.)

Lee proves that $Kh'(K) = H_*(CKh(K), d')$ is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$, and is generated by two canonical cycles which correspond to the two possible orientations of K . Represent the transverse knot by an oriented k -braid, and let \mathfrak{s}_o denote the corresponding canonical cycle. The oriented resolution of this braid is a union of k nested circles, and

$$\mathfrak{s}_o = (\mathbf{v}_- + (-1)^d \mathbf{v}_+) \otimes (\mathbf{v}_- + (-1)^{d+1} \mathbf{v}_+) \otimes \cdots \otimes (\mathbf{v}_- + (-1)^{d+k-1} \mathbf{v}_+)$$

is the element of $CKh(K)$ obtained by alternately labeling these circles by $\mathbf{v}_- + \mathbf{v}_+$ and $\mathbf{v}_- - \mathbf{v}_+$ (the label on the outermost circle depends on the orientation of the knot). Recall that $\psi(K) = [\tilde{\psi}(K)] \in Kh(K)$, where

$$\tilde{\psi}(K) = \mathbf{v}_- \otimes \mathbf{v}_- \otimes \cdots \otimes \mathbf{v}_-$$

is the q -homogeneous part of \mathfrak{s}_o with the lowest q -grading [32].

In [33], Rasmussen defines a function s on $Kh'(K)$ whose value on $x \in Kh'(K)$ is the largest n such that x can be represented by a cycle whose q -homogeneous terms all have q -gradings at least n . The invariant $s(K)$ is then defined so that $s(K) - 1 = s_{min} = s([\mathfrak{s}_o])$. Recall from [32] that $q(\tilde{\psi}(K)) = sl(K)$. Therefore, the hypothesis of Theorem 1.2 implies that $q(\tilde{\psi}(K)) = s(K) - 1$. If $\psi(K) = 0$, then $\tilde{\psi}(K) = dy$ for some $y \in CKh(K)$. Since d preserves quantum gradings, it must be that $q(y) = s(K) - 1$ as well. Then $\mathfrak{s}_o - d'y$ is a cycle in $(CKh(K), d')$ which is homologous to \mathfrak{s}_o and whose q -homogeneous terms all have q -gradings strictly greater than $s(K) - 1$. This contradicts the equality $s([\mathfrak{s}_o]) = s(K) - 1$.

To make a similar argument with \mathbb{Z}_2 coefficients, we must use Turner's modification of Lee's construction (since $Kh'(K)$ is isomorphic to $Kh(K)$ over \mathbb{Z}_2). In [38], Turner defines a differential d'' on $CKh(K)$ (now, with \mathbb{Z}_2 coefficients) using the multiplication and comultiplication maps given by

$$\begin{aligned} m(\mathbf{v}_+ \otimes \mathbf{v}_+) &= \mathbf{v}_+ & \Delta(\mathbf{v}_+) &= \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ + \mathbf{v}_+ \otimes \mathbf{v}_+ \\ m(\mathbf{v}_- \otimes \mathbf{v}_-) &= \mathbf{v}_- & \Delta(\mathbf{v}_-) &= \mathbf{v}_- \otimes \mathbf{v}_- \\ m(\mathbf{v}_+ \otimes \mathbf{v}_-) &= m(\mathbf{v}_- \otimes \mathbf{v}_+) = \mathbf{v}_-. \end{aligned}$$

He shows that $Kh''(K) = H_*(CKh(K), d'')$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and is generated by two cycles corresponding to the two orientations of K . If we represent K by an oriented braid, then the corresponding cycle \mathfrak{s}_o is obtained by alternately labeling the nested components of the oriented resolution by $\mathbf{v}_- + \mathbf{v}_+$ and \mathbf{v}_- (as before, the label on the outermost circle depends on the orientation of the knot). The transverse invariant $\psi(K)$ is again the image in $Kh(K)$ of the q -homogeneous part of \mathfrak{s}_o with the lowest q -grading. Moreover, an analogue of the function s can be defined in this setting, and it takes the same values as Rasmussen's s function [20]. Therefore, our argument from the preceding paragraph still applies.

It remains to deal with the reduced case. Mark a point on K , and let $CKh_-(K)$ denote the subcomplex generated by elements in which the marked circle is labeled by \mathbf{v}_- in every resolution. The reduced Khovanov chain complex is the quotient complex

$$\widetilde{CKh}(K) = CKh(K)/CKh_-(K).$$

Turner's construction works just as well for reduced Khovanov homology over \mathbb{Z}_2 . In this case, $\widetilde{Kh}''(K) = H_*(\widetilde{CKh}(K), d'')$ is isomorphic to \mathbb{Z}_2 . When K is a braid, $\widetilde{Kh}''(K)$ is generated by the element \mathfrak{s}_o , which is obtained by alternately labeling the nested components of the oriented resolution of K by $\mathbf{v}_- + \mathbf{v}_+$ and \mathbf{v}_- so that the marked circle is labeled by $\mathbf{v}_- + \mathbf{v}_+$. The q -homogeneous part of \mathfrak{s}_o with the lowest q -grading is the element $\tilde{\psi}(K)$ formed by labeling the marked circle in the oriented resolution of K by \mathbf{v}_+ , and every other circle by \mathbf{v}_- . The quantum grading is shifted by 1 in the reduced theory, so that the analogue \tilde{s} of Rasmussen's function s satisfies $\tilde{s}([\mathfrak{s}_o]) = s(K)$. Since $q(\tilde{\psi}(K)) = sl(K) + 1$ in the reduced theory, we may proceed as before to show that $sl(K) = s(K) - 1$ implies that $\psi(K) \neq 0$.

To prove the converse, note that if $\widetilde{Kh}(K)$ is thin, then the component $\widetilde{Kh}^{0,*}(K)$ is non-trivial only in the quantum grading $q = s(K)$. Since $\psi(K) \in \widetilde{Kh}^{0,*}(K)$, and $q(\psi(K)) = sl(K) + 1$, it follows that if $\psi(K) \neq 0$, then $sl(K) = s(K) - 1$. □

Proof of Corollary 1.3. This follows immediately from Theorem 1.2 and Theorem 1.1, together with the fact that if K is quasi-alternating, then the spectral sequence from $\widetilde{Kh}(K)$ to $\widehat{HF}(-\Sigma(K))$ collapses at the E^2 term and $s(K) = \sigma(K)$ [21]. □

7. EXAMPLES

7.1. Tightness by means of Khovanov homology. In this subsection, we give examples of contact structures whose tightness can be established by means of Khovanov homology. We first focus on the case of double covers of transverse braids for which Corollary 1.3 applies. When K is a quasipositive braid, the equality $sl(K) = s(K) - 1$ holds, but the contact structure ξ_K is Stein-fillable and, therefore, automatically tight. To get more interesting examples, we look for non-quasipositive knots for which $sl = s - 1$. The mirrors of 10_{125} , 10_{130} , and 10_{141} are the only such knots with 10 crossings or fewer. (As was indicated to the second author by Lenny Ng, this may be verified by contrasting the list of quasipositive knots from [2] and the values of the maximal self-linking numbers [22].) We use each of these knots to obtain an infinite family of tight contact structures, and we show that the contact structures in these families are not Stein-fillable.

We also use Theorem 1.1 to give examples of tight contact 3-manifolds which do not obviously arise as branched double covers of (S^3, ξ_{st}) along transverse links. Unfortunately, we do not have any analogues of Theorem 1.2 and Corollary 1.3 in this more general situation. Instead, we use the computer programs **Kh** and **Trans** [3] to check the collapsing of the spectral sequence and non-vanishing of ψ .

It will be helpful to use contact surgery presentations for the contact structures we consider. The idea [1, 31] in the construction of such surgery diagrams is to isotope a page of the open book and certain curves on that page so that all curves on which Dehn twists are performed become Legendrian knots. Right-handed Dehn twists are then equivalent to Legendrian surgeries, and left-handed Dehn twists to $(+1)$ contact surgeries. We can (and will) always assume that the monodromy presentation of an open book starts with a sequence of right-handed Dehn twists whose product yields an open book compatible with (S^3, ξ_{st}) ; this allows us to get rid of 1-handles and perform all the contact surgeries on Legendrian knots in S^3 . More precisely, a sequence of Dehn twists performed in a certain order corresponds to a sequence of surgeries on push-offs of the corresponding curves. Some care is needed to determine the linking of these push-offs; we refer the reader to [12] for details (in the case of branched covers), and only state the answer for the more general case that we need here. The procedure of “Legendrianizing” an open book is illustrated on Figure 13. A page of an open book for S^3 with the monodromy given by $D_{\alpha_2} D_{\alpha_3} \dots D_{\alpha_n}$ can be visualized as the Seifert surface S of a $(2, n)$ torus knot; the curves $\alpha_1, \dots, \alpha_n$ become loops around subsequent Hopf bands forming the surface S , and α_0 is a somewhat more complicated curve (see top of Figure 13). Next, the torus knot can be placed into S^3 (with the standard contact form $dz - y dx$) so that S becomes a page of an open book compatible with the contact structure; the curves $\alpha_0, \dots, \alpha_n$ are now Legendrian knots that can be seen on this page (bottom of Figure 13). (Some care must be taken to place two strands of α_0 on the same thin strip; this can be achieved by slightly twisting the strip.)

Considering various push-offs of these curves (shown on Figure 14, see also [12]) step-by-step, we can obtain surgery diagrams corresponding to arbitrary monodromies. We observe that the curves $\alpha_1, \dots, \alpha_n$ (and their push-offs) yield Legendrian unknots with $tb = -1$, so the diagrams for branched double covers all consist of standard Legendrian unknots only; the curve α_0 gives a stabilized Legendrian unknot with $tb = -2$.

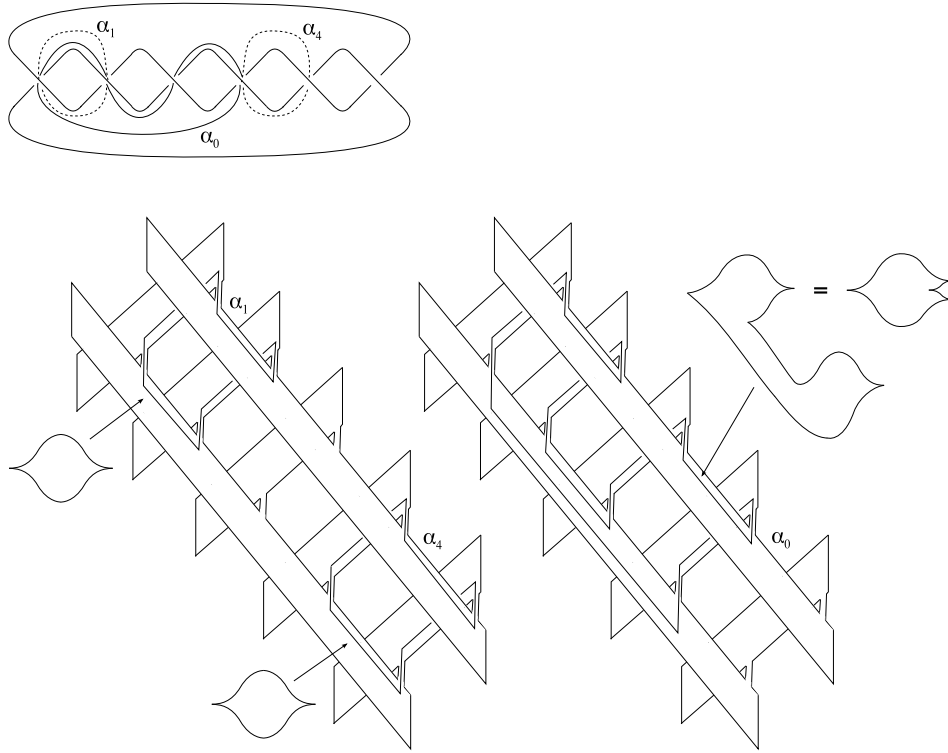
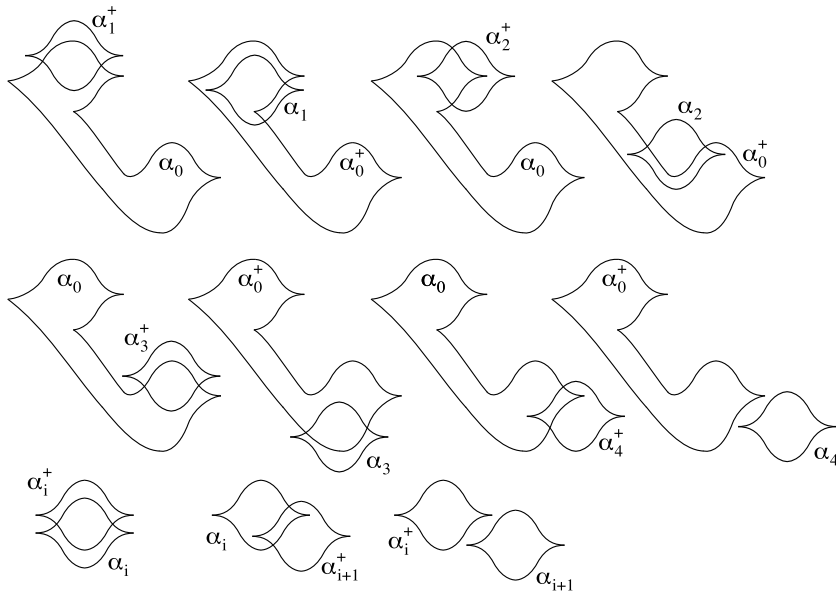


FIGURE 13. From open books to surgery diagrams.

FIGURE 14. The linking of Legendrian push-offs of the curves α_i .

Example 7.1. For $r \geq 5$, consider the pretzel link $P(-r, 3, -2)$ (for $n = 5$, this is the mirror of the knot 10_{125}), and let K_r be its transverse representative given by the closed braid

$$(\sigma_1)^{-r} \sigma_2 \sigma_1^3 \sigma_2.$$

We use the algorithm described above to obtain the contact surgery description for the induced contact structure ξ on the branched double cover $\Sigma(K_r)$. The resulting surgery diagram for $r = 5$ is shown on the right of Figure 15 (when $r > 5$, we get the diagram with r (+1)-surgeries). The unoriented surgery link is Legendrian isotopic to the one shown on the left of Figure 15, so we can work with the more symmetric diagram. In other examples of this section, we will similarly change surgery links by Legendrian isotopy to make pictures more symmetric.

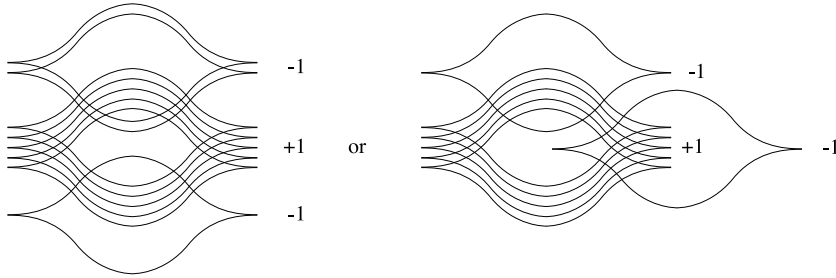


FIGURE 15. The surgery diagram for the branched double cover of the transverse knot $K = K_5$ in Example 7.1 .

The underlying smooth manifold $\Sigma(K_r)$ is the Seifert fibered space $M(-1; 2/3, 1/2, 1/n)$. (See Figure 17 for a sequence of Kirby calculus moves demonstrating this for $n = 5$.) Each link K_r is quasi-alternating. Indeed, $|\det(K_r)| = |H_1(M(-1; 2/3, 1/2, 1/n))| = r + 6$. On the other hand, resolving the crossing circled in Figure 16 in two possible ways, we obtain the link K_{r-1} and the unknot. Repeating the procedure r times, we get the link K_0 , which is the trefoil linked once with the unknot. Thus K_0 is an alternating link with $|\det(K_0)| = 6$, and, since $|\det(K_r)| = |\det(K_{r-1})| + |\det(\text{unknot})|$, we see by induction that K_r is quasi-alternating.

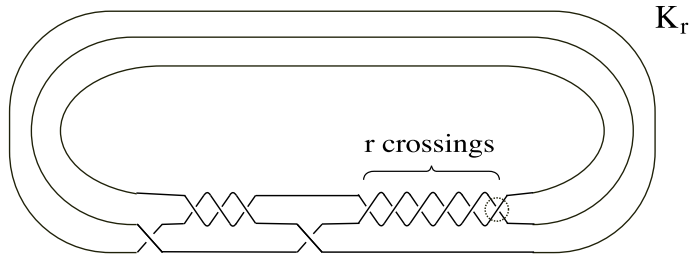


FIGURE 16. The links K_r are quasi-alternating.

We next check the hypothesis of Theorem 1.2 when r is odd (i.e. K_r is a knot). We compute $sl(K) = 2 - r$. Since the knot K_r is quasi-alternating, s equals to the signature $\sigma(K_r) = 3 - r$

(we compute the signature via the Goeritz matrix of the knot [11]). When r is even, K_r is a two-component link, so Theorem 1.2 does not apply. However, we can argue that $\psi(K_r) \neq 0$ by [32, Theorem 4], since $\psi(K_{r-1}) \neq 0$, and the transverse braid K_{r-1} is obtained from K_r by resolving a negative crossing.

Corollary 1.3 implies that the branched double cover of each K_r is a tight contact manifold. We now show that none of them are Stein-fillable. Since $\Sigma(K_5)$ can be obtained from any of $\Sigma(K_r)$ by a sequence of Legendrian surgeries, it suffices to check that $\xi = \xi_{K_5}$ (shown on Figure 15) is not Stein-fillable.

First, we compute the homotopy invariants of the contact structure ξ . Recall [7] that the three-dimensional invariant d_3 of a contact structure given by a contact surgery diagram can be computed as

$$d_3(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X) + 2}{4} + m,$$

where X is a 4-manifold bounded by Y and obtained by adding 2-handles to B^4 as dictated by the surgery diagram, \mathfrak{s} is the corresponding Spin^c structure on X , and m is the number of $(+1)$ -surgeries in the diagram. The Spin^c structure \mathfrak{s} arises from an almost-complex structure defined in the complement of a finite set of points in X , and the class $c_1(\mathfrak{s})$ evaluates on each homology generator of X corresponding to the handle attachment along an (oriented) Legendrian knot as the rotation number of the knot. Thus, for the contact structure ξ on $Y = M(-1; 2/3, 1/2, 1/5)$ defined by the surgery diagram from Figure 15, we compute $c_1(\mathfrak{s}) = 0$ and $d_3(\xi) = -\frac{1}{2}$.

We show that ξ is not Stein-fillable, combining the ideas from [9] and [19]. More precisely, we will show that Y carries no Stein-fillable contact structures with $d_3 = -\frac{1}{2}$. Note that Y is an L -space since it is the branched double cover of a quasi-alternating knot [30]. Suppose that (X, J) is a Stein-filling for ξ , and \mathfrak{s}_J is the corresponding Spin^c structure on X . Note that \mathfrak{s}_J restricts to the Spin^c structure \mathfrak{s}_ξ on Y , and $c_1(\mathfrak{s}_\xi) = 0$. Let $\bar{\xi}$ be the contact structure on Y conjugate to ξ ; then $\mathfrak{s}_{\bar{\xi}} = \mathfrak{s}_\xi$, and $\bar{\xi}$ has a Stein-filling $(X, -J)$ with $\mathfrak{s}_{-J} = \mathfrak{s}_J$. We claim that $c_1(\mathfrak{s}_J) = 0$. Indeed, otherwise the Spin^c structures \mathfrak{s}_J and \mathfrak{s}_{-J} are not isomorphic, and by [31] the contact elements $c(\xi)$ and $c(\bar{\xi})$ are linearly independent elements in homology $\widehat{HF}(-Y, \mathfrak{s}_\xi)$. But Y is an L -space, so $\widehat{HF}(-Y, \mathfrak{s}_\xi)$ has rank 1, a contradiction.

On the other hand, the fact that Y is an L -space implies [25] that $b_2^+(X) = 0$ for any symplectic filling X of a contact structure on Y . By the argument in [9], it follows that $b_1(X) = 0$. Now, observe that the space $-Y$ can be represented as the boundary of the plumbing shown on Figure 18.

Denote by W the 4-manifold with boundary $-Y$ given by this plumbing. If X is a symplectic filling for ξ , then $X \cup W$ is an oriented negative-definite closed 4-manifold. By Donaldson's theorem, the intersection form on $X \cup W$ is standard diagonal $\langle -1 \rangle^n$. To get restrictions on the intersection form of X , we consider the embeddings of the lattice given by Figure 18 into the standard negative-definite lattice, following [19]. Let e_i , $i = 1, 2, \dots, n$, be the basis of $\langle -1 \rangle^n$ such that $e_i \cdot e_j = -\delta_{ij}$. Let v_i be the basis of $H_2(W)$ corresponding to the vertices of the plumbing graph of Figure 18. Up to permutations and sign reversals of e_i (which are

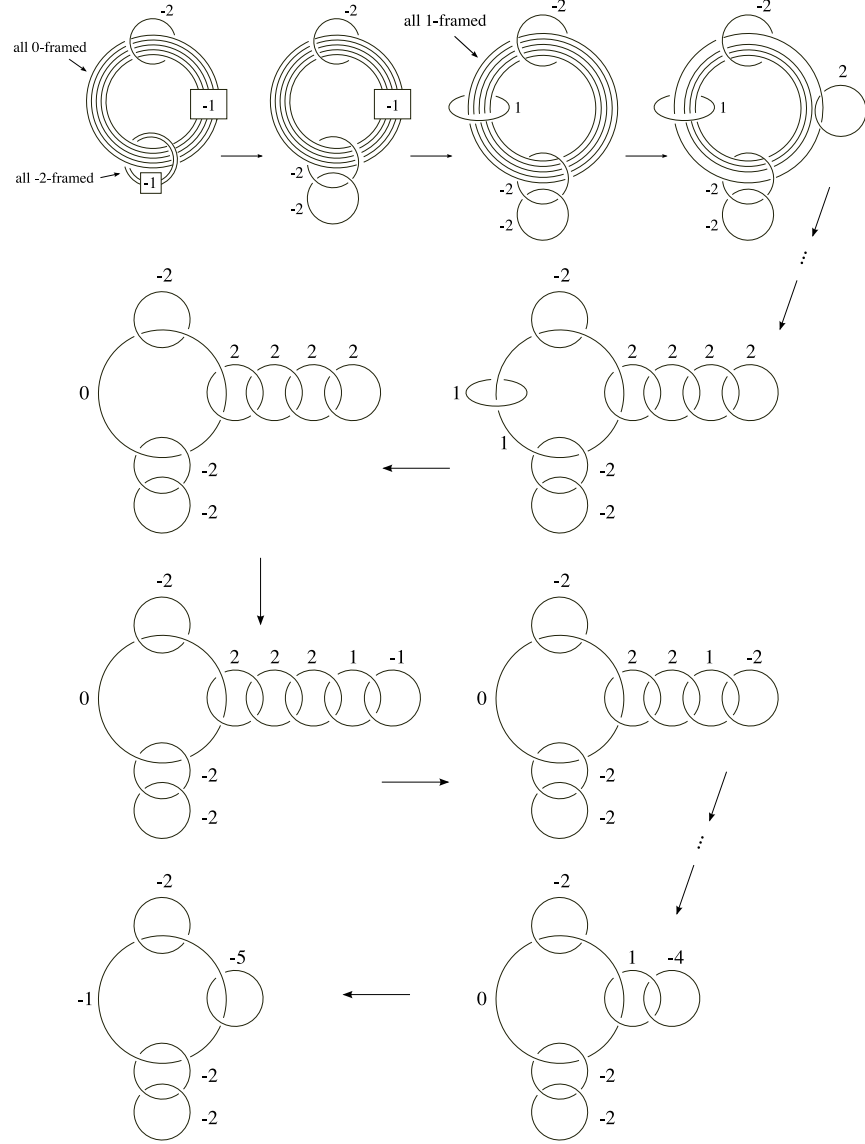


FIGURE 17. Kirby moves.

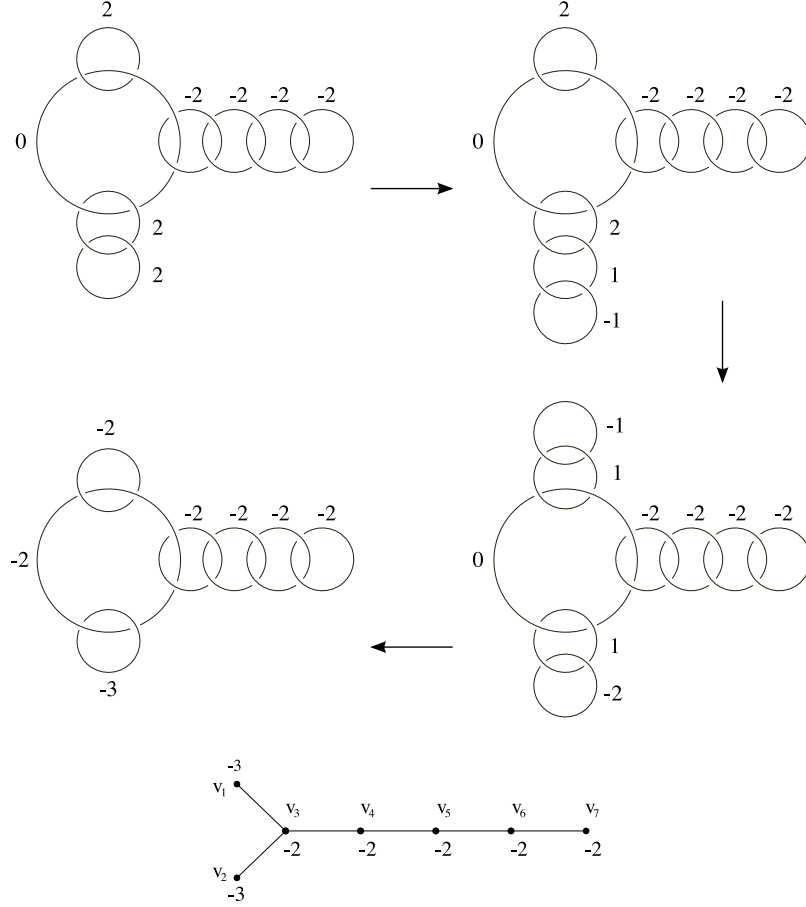
automorphisms of the lattice $\langle -1 \rangle^n$, we have

$$\begin{aligned} v_3 &\mapsto e_1 + e_2, & v_2 &\mapsto -e_1 + e_3, & v_1 &\mapsto -e_1 - e_3 + e_5, \\ v_4 &\mapsto -e_2 + e_4, & v_5 &\mapsto -e_4 + e_6, & v_6 &\mapsto -e_6 + e_7, & v_7 &\mapsto -e_7 + e_8 \end{aligned}$$

(Another possibility would be for the first four vectors to embed as

$$v_3 \mapsto e_1 + e_2, \quad v_2 \mapsto -e_1 + e_3, \quad v_1 \mapsto -e_2 + e_4 + e_5, \quad v_4 \mapsto -e_1 - e_3,$$

but this leads to a contradiction when we try to embed v_5 .)

FIGURE 18. Kirby diagrams for $-Y$ and the plumbing graph for W .

The orthogonal complement L of the image of the lattice generated by images of v_i 's in $\langle -1 \rangle^n$ is then spanned by the vectors

$$-e_1 + e_2 - e_3 + e_4 - 2e_5 + e_6 + e_7 + e_8, \quad e_9, \quad \dots \quad e_n,$$

and the intersection form on L is the diagonal form $\langle -11 \rangle \oplus \langle -1 \rangle^{n-8}$. Because $H_1(Y) = \mathbb{Z}/11$ (indeed, $|H_1(Y)| = \det(10_{125}) = 11$), and both $H_2(X)$, $H_2(W)$ are torsion-free, we have

$$0 \rightarrow H_2(X) \oplus H_2(W) \rightarrow H_2(X \cup W) \rightarrow \mathbb{Z}/11 \rightarrow 0,$$

and thus $H_2(X, \mathbb{Z})$ is a subgroup of $L = \mathbb{Z}^{n-7}$ of index 11. Set $m = n - 7 = b_2(X)$, and let $\{u_1, u_2, \dots, u_m\}$ be basis of L in which the form is diagonal, and $u_1 \cdot u_1 = -11$. The vectors $11u_1, 11u_2, \dots, 11u_m$ lie in $H_2(X, \mathbb{Z})$, and generate $H_2(X, \mathbb{Q})$ over \mathbb{Q} .

It follows that $c_1(\mathfrak{s}_J)$ must evaluate as an odd integer on each vector $11u_1, 11u_2, \dots, 11u_m$; but since $c_1(\mathfrak{s}_J) = 0$, this means that $m = 0$. Then $d_3(\xi) = 0$, which contradicts the calculation $d_3(\xi) = -\frac{1}{2}$.

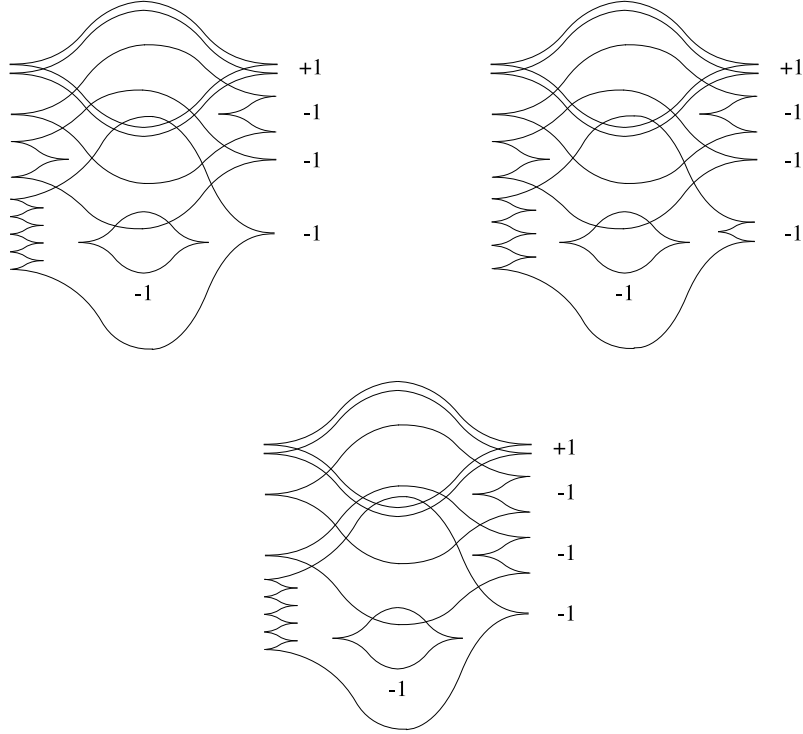


FIGURE 19. The manifold $Y = M(-1; 2/3, 1/2, 1/5)$ carries three tight contact structures: ξ_1 (top left), ξ_2 (top right), and Ξ (bottom).

Remark 7.2. The branched double cover of the knot K in the previous example is the Seifert fibered space $Y = M(-1; 2/3, 1/2, 1/5)$. Tight contact structures on this space were classified in [9]: Y carries three tight contact structures ξ_1 , ξ_2 and Ξ given by surgery diagrams on Figure 19. Of these, ξ_1 and ξ_2 are known to be Stein-fillable; we can thus conclude that our contact structure ξ is isotopic to Ξ .

Example 7.3. Consider the transverse representative of the mirror of the knot 10_{141} given by the braid $\sigma_1^{-4}\sigma_2\sigma_1^3\sigma_2^2$. We consider the family of braids

$$K_r = \sigma_1^{-r}\sigma_2\sigma_1^3\sigma_2^2.$$

The contact surgery description for the corresponding contact structures is given on Figure 20; the surgery diagrams are quite similar to those in the previous example, but have one extra component. The Kirby calculus moves similar to those in Figure 17 show that the branched double cover is the Seifert fibered space $M(-1; 2/3, 2/3, 1/n)$.

As before, we can show that all the braids K_r are quasi-alternating. Indeed, we resolve one of the negative crossings to obtain K_{r-1} and a trefoil as two resolutions; we also observe that K_0 is the connected sum of two trefoils. Since $|\det(\text{trefoil})| = 3$, $|\det(K_0)| = 9$ and $|\det(K_r)| = |H_1(M(-1; 2/3, 1/2, 1/n))| = 9 + 3r$, each K_r is quasi-alternating by induction.

Next, we compute $sl(K_r) = 3 - r$, and $s = \sigma(K_r) = 4 - r$; the hypotheses of Corollary 1.3 are therefore satisfied, and all branched covers $\Sigma(K_r)$ are tight contact manifolds.

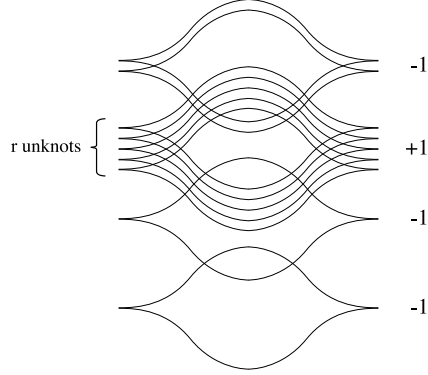


FIGURE 20. The surgery diagrams for the branched double covers of the transverse links K_r in Example 7.3.

For the contact structure on the branched cover of K_4 , we compute $d_3 = 0$, which provides no obstruction to Stein-fillability. However, for the braid K_6 we get $d_3 = -\frac{1}{2}$. We then argue as in the previous example to show that the branched cover of K_6 is not Stein-fillable (and thus the branched double covers of all braids K_r with $r \geq 6$ are not Stein-fillable either). Denote $Y = \Sigma(K_6) = M(-1; 2/3, 1/2, 1/6)$; then $-Y$ is the boundary of the plumbing W

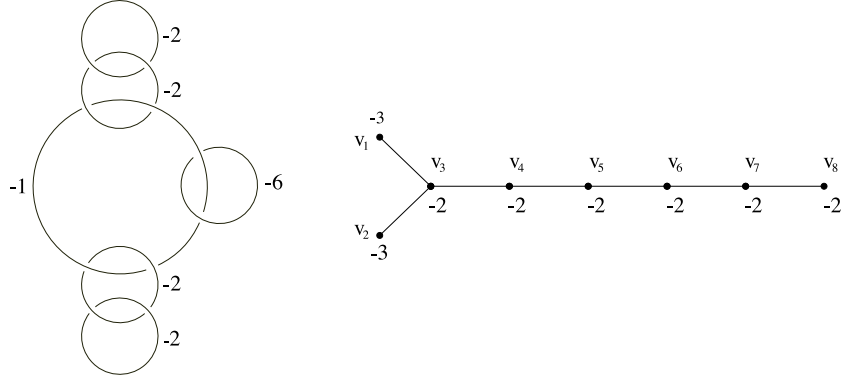


FIGURE 21. A Kirby diagram for $Y = \Sigma(K_6)$ and the plumbing graph for $-Y$ of Example 7.3.

given by the graph on Figure 21. As before, for any symplectic filling X of Y the union $X \cup W$ is a negative-definite closed 4-manifold with the standard diagonal intersection form. Up to changing the signs and the order of the vectors e_i in the diagonal basis, there is a unique embedding of the lattice given by Figure 21 into $\langle -1 \rangle^n$, given by

$$\begin{array}{llll} v_3 \mapsto e_1 + e_2, & v_1 \mapsto -e_1 - e_3 + e_5, & v_2 \mapsto -e_1 + e_3 + e_4, & v_4 \mapsto -e_2 + e_6, \\ v_5 \mapsto -e_6 + e_7, & v_6 \mapsto -e_7 + e_8, & v_7 \mapsto -e_8 + e_9, & v_8 \mapsto -e_9 + e_{10}, \end{array}$$

and thus the orthogonal complement of this lattice in $\langle -1 \rangle^n$ is $\langle -9 \rangle \oplus \langle -3 \rangle \oplus \langle -1 \rangle^{n-10}$. As in the previous example, we must have $c_1(X) = 0$ for any Stein-filling. Since $|H_1(Y)| = 27$, similar parity argument shows that $b_2(X) = 0$, and so d_3 must be zero, a contradiction.

Remark 7.4. One can try to argue as in [9] to investigate symplectic fillability in Examples 7.1 and 7.3: a slightly more involved argument modulo 8 puts further restrictions on the value d_3 for symplectic fillings (with diagonal odd intersection form). However, this gives no obstruction to symplectic fillability of any contact structures in the above two examples.

In the opposite direction, certain tight open books with the punctured torus page and pseudo-Anosov monodromy can be shown to be symplectically fillable as perturbations of taut foliations [14]. We note that our examples are not pseudo-Anosov, so these results do not apply.

Example 7.5. A transverse representative of the mirror of 10_{130} with the maximal self-linking number is given by the braid $\sigma_1^{-3}\sigma_2\sigma_1^2\sigma_2^2\sigma_3\sigma_2^{-1}\sigma_3$. We consider a family of transverse braids

$$K_r = \sigma_1^{-r}\sigma_2\sigma_1^2\sigma_2^2\sigma_3\sigma_2^{-1}\sigma_3.$$

First, we check that all the underlying links are quasi-alternating. Resolve of the negative crossings among those given by σ_1^{-r} to obtain K_{r-1} as one of the resolutions and the unknot as the other. Observe that K_0 is a two-component alternating link of $\det = 14$ (with 5_2 knot and the unknot as components, linked once). Finally, compute $|\det(K_r)| = 14 + r$ (one way to see this is to compute the size of H_1 of the branched double cover of K_r which is a Seifert fibered space shown on Figure 22).

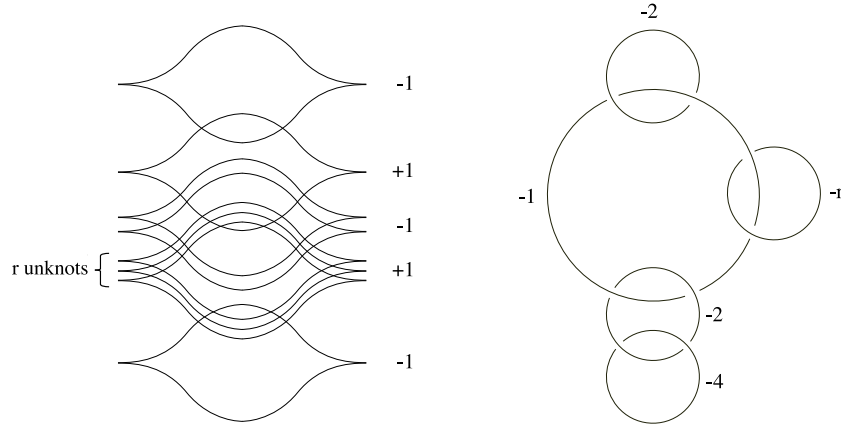


FIGURE 22. The surgery diagrams for the branched double covers of the transverse links K_r in Example 7.5.

The hypothesis of Corollary 1.3 holds: $sl(K_r) = 2 - r = \sigma - 1$. Therefore, the branched double covers of the transverse links K_r are all tight.

These contact structures are also Stein non-fillable for $r \geq 5$; this can be seen by using the arguments of [6, Section 7]. (Although Theorem 7.1 of [6] is stated for the case of 3-braids, its proof carries over to our example, since the corresponding 3-manifold is an L -space, the

contact structure is self-conjugate, and its 3-dimensional invariant is negative. We leave the details to the reader.)

Remark 7.6. In Examples 7.1 and 7.3, transverse links are 3-braids, and the contact structures on the branched double covers can be given by open books whose page is a once-punctured torus. Tightness of these contact structures can be established by using results of [13] or [4]. However, these results do not apply to Example 7.5 as it deals with 4-braids, and the page of the corresponding open books is a twice-punctured torus.

In all of the above examples, we checked explicitly that our families of links are quasi-alternating. In fact, a more involved argument (see [6]) shows that all braids of the form $(\sigma_1\sigma_2)^3\sigma_1\sigma_2^{-a_1}\dots\sigma_1\sigma_2^{-a_n}$ (with $a_i \geq 0$) are quasi-alternating and have $sl = \sigma - 1$. Thus Corollary 1.3 applies to many more knots; moreover, a lot of the corresponding contact manifolds are not Stein-fillable.

We also note that a weaker condition, $\text{rk}(\widetilde{Kh}(K)) = |\det(K)|$ is sufficient to ensure that the spectral sequence from \widetilde{Kh} to \widehat{HF} collapses at the E^2 stage. This can be checked (for any individual reasonably small knot) by a computer, for example using Kh program that computes the rank of reduced Khovanov homology with \mathbb{Z}_2 coefficients. Checking the second condition, $sl(K) = s - 1$, is also routine for \widetilde{Kh} -thin knots (alternatively, one can use the Trans program to check $\psi \neq 0$). Proving tightness of the contact structure on the branched double cover is thus reduced to a computer calculation.

In the next example, we prove tightness of a contact structure which does not obviously arise as the branched cover of (S^3, ξ_{st}) along a transverse link (the corresponding open book includes a Dehn twist around α_0).

Example 7.7. Consider the contact manifold with surgery diagram shown on Figure 23. This diagram corresponds (via the procedure described in the beginning of this section together with a Legendrian “flip” as in Example 7.1) to the open book whose page S is a genus 2 surface with one boundary component, and the monodromy is $\phi = D_{\alpha_1}D_{\alpha_2}D_{\alpha_3}D_{\alpha_4}(D_{\alpha_3})^{-5}(D_{\alpha_4})^2D_{\alpha_0}$ (where D_{α_1} is performed first and D_{α_0} last). Using Trans and Kh programs, we find that $\psi(S, \psi) \neq 0$ and that $\text{rk}(\widetilde{Kh}(S, \phi)) = 9$. Since $\text{rk}(H_1(Y)) = 9$ as well (we compute it as the determinant of the linking matrix), the spectral sequence collapses at the E^2 term. Therefore, Figure 7.7 represents a tight contact structure. (A few Kirby moves as above show that the underlying smooth manifold is the Seifert fibered space $Y = M(-1; 2/3, 1/3, 1/5)$).

7.2. Some limitations. The examples in this subsection illustrate some of the limitations of our method.

Example 7.8. Consider a family of (non-quasi-alternating) transverse 3-braids

$$K_r = \sigma_1^{-r}\sigma_2\sigma_1^2\sigma_2.$$

Using the Trans program and [32, Theorem 4], we see that the transverse invariant ψ vanishes for $r > 2$. However, the calculations of the contact invariant in [13] imply that the corresponding contact structures on the branched double covers are all tight, and have $c(\xi) \neq 0$. Thus, Khovanov homology fails to detect tightness in this case. It is interesting to take a closer look at the corresponding contact structures: they are given by surgery diagrams on the left of Figure 24, and are very similar to the contact structures from Example 7.1. As

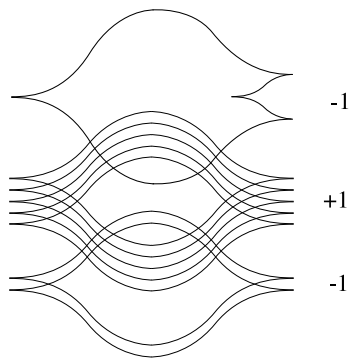


FIGURE 23. A surgery diagram for the tight contact structure in Example 7.7.

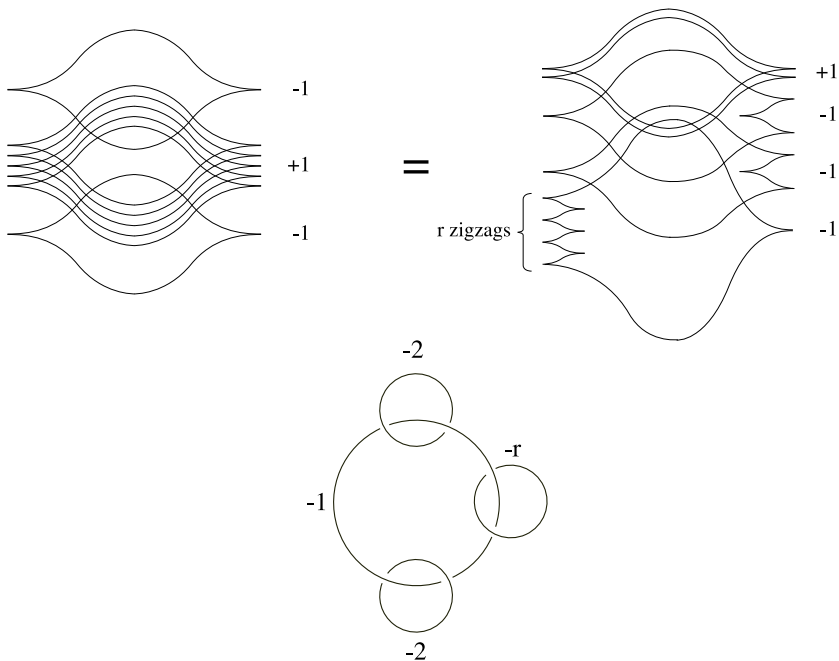


FIGURE 24. The surgery diagrams for the branched double covers of the transverse links in Example 7.8.

the latter are obtained from the former by Legendrian surgery on a knot, the contact structures ξ_{K_r} cannot be Stein-fillable. The underlying smooth manifold is $M(-1; 1/2, 1/2, 1/r)$; by [9] it carries a unique tight, Stein non-fillable contact structure for each r (shown on the right of Figure 24), and we conclude that the contact structures ξ_{K_r} are precisely those considered in [9]. By [9], most of these contact structures are not symplectically fillable, and one may wonder whether there is a relationship between the vanishing of ψ and symplectic non-fillability which goes beyond Proposition 1.4 (although such a relationship seems quite improbable).

Example 7.9. We find it instructive to give an example where a non-vanishing element $\psi \in \widehat{Kh}(S, \phi)$ dies in the spectral sequence, so that $c(\xi) = 0$. Consider the open book with once-punctured genus two page S , and the monodromy $\phi = D_{\alpha_1} D_{\alpha_2} D_{\alpha_3} D_{\alpha_4} (D_{\alpha_3})^{-5} D_{\alpha_4} D_{\alpha_0}$; the corresponding contact manifold is given by the surgery diagram on Figure 25. By Kirby calculus similar to Figures 17, 18, the underlying manifold is the boundary of the E_8 plumbing, i.e. the Poincaré homology sphere $-\Sigma(2, 3, 5)$ with orientation reversed; by [8], it carries no tight contact structures, therefore $c(\xi) = 0$. However, **Trans** tells us that $\psi(S, \phi) \neq 0$. (In this case we have $\text{rk}(\widehat{HF}) = 1$, while the E^2 term of the spectral sequence is of course bigger: **Kh** computes $\text{rk}(\widehat{Kh}(S, \phi)) = 7$.) This example shows that the vanishing of $\psi(S, \phi)$ is not an invariant of $\xi_{S, \phi}$, as discussed at the end of Section 5.

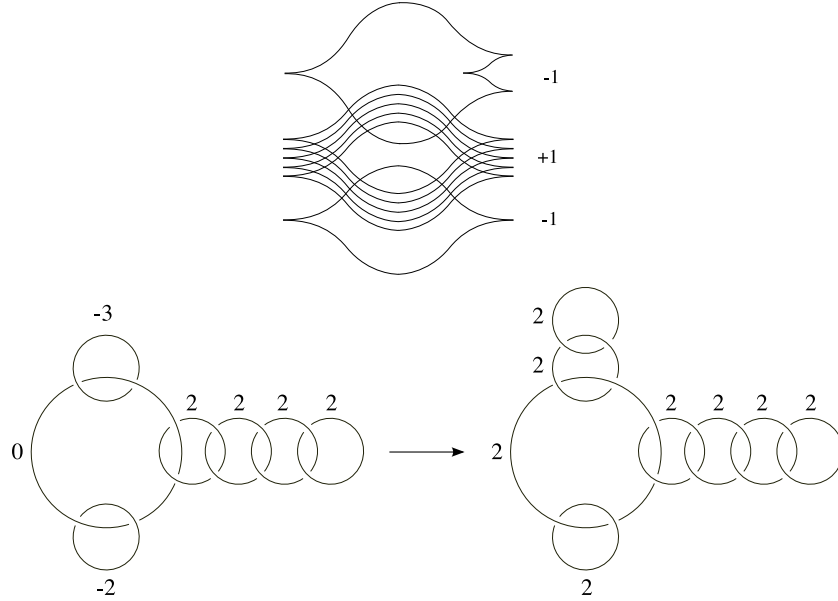


FIGURE 25. A surgery diagram for the contact structure in Example 7.9.

7.3. Non-fillability by means of Khovanov homology. In this section, we give two examples which illustrate the use of Proposition 1.4 in proving that a contact structure is not strongly symplectically fillable.

Example 7.10. Let $m(10_{132})$ denote the mirror of the knot 10_{132} . The Poincaré polynomial for the reduced Khovanov homology of $m(10_{132})$ is

$$t^{-7}q^{-14} + t^{-6}q^{-12} + t^{-5}q^{-10} + 2t^{-4}q^{-8} + t^{-3}q^{-8} + t^{-3}q^{-6} + t^{-2}q^{-6} + t^{-2}q^{-4} + t^{-1}q^{-2} + t^0q^{-2}.$$

(Here, the exponent of t keeps track of the homological grading, while the exponent of q keeps track of the quantum grading.) Note that the reduced Khovanov homology is supported in non-positive homological gradings. Therefore, Corollary 1.5 implies that if K is any transverse knot which is smoothly isotopic to $m(10_{132})$, and $sl(K) + 1 \neq -2$, then $c(\xi_K) = 0$, and, hence,

ξ_K is not strongly symplectically fillable (or weakly symplectically fillable, since $\Sigma(K)$ is a rational homology 3-sphere³). For example, the 4-braid

$$\sigma_3^{-1}\sigma_2^2\sigma_3^{-2}\sigma_2^{-1}\sigma_3\sigma_1\sigma_2^{-1}\sigma_1^{-2}$$

corresponds to a transverse representative K of $m(10_{132})$ (this braid representative comes from [16]) with $sl(K) + 1 = -6$; hence, $c(\xi_K) = 0$.

Remark 7.11. It is not hard to find transverse knots which satisfy the hypotheses of Corollary 1.5. For instance, of the 49 prime knots with 9 crossings, the knots (or their mirrors) in the set below have reduced Khovanov homologies supported in non-positive homological gradings:

$$\{9_1, 9_2, 9_3, 9_4, 9_5, 9_6, 9_7, 9_9, 9_{10}, 9_{13}, 9_{16}, 9_{18}, 9_{23}, 9_{35}, 9_{38}, 9_{45}, 9_{46}, 9_{49}\}$$

In the next example, we prove non-fillability of a contact structure which does not obviously arise as the branched cover of (S^3, ξ_{st}) along a transverse link (the corresponding open book includes a Dehn twist around α_0).

Example 7.12. Let (S, ϕ) be the abstract open book

$$(S_{2,1}, D_{\alpha_1}^2 D_{\alpha_2}^{-1} D_{\alpha_3}^{-1} D_{\alpha_4}^{-1} D_{\alpha_0} D_{\alpha_4}^{-1} D_{\alpha_3}^{-1} D_{\alpha_2}^{-1} D_{\alpha_1}^{-1} D_{\alpha_3} D_{\alpha_4} D_{\alpha_2}).$$

Using the program `Kh`, we find that the Poincaré polynomial for $\widetilde{Kh}(S, \phi)$ is

$$t^{-6} + 2t^{-5} + t^{-4} + t^{-3} + 2t^{-2} + 2t^{-1}.$$

(Here, the exponent of t keeps track of the homological grading.) Note that $\widetilde{Kh}(S, \phi)$ is supported in negative homological gradings. Since the homological grading of $\psi(S, \phi)$ is zero, it follows at once that $\psi(S, \phi)$ vanishes. Therefore, Proposition 1.4 implies that $c(S, \phi) = 0$, and, hence, that $\xi_{S, \phi}$ is not strongly symplectically fillable.

³Strong symplectic fillability of (M, ξ) is equivalent to weak symplectic fillability when M is a rational homology 3-sphere [23].

8. COMULTIPLICATION

In [5], the first author shows that the contact invariant $c(S, \phi)$ satisfies the following naturality property with respect to a comultiplication map on Heegaard Floer homology.

Theorem 8.1 ([5, Theorem 1.4]). *If h and g are two boundary-fixing diffeomorphisms of S , then there is a comultiplication map*

$$\mu : \widehat{HF}(-M_{S,hg}) \rightarrow \widehat{HF}(-M_{S,h}) \otimes_{\mathbb{Z}_2} \widehat{HF}(-M_{S,g})$$

which sends $c(S, hg)$ to $c(S, h) \otimes c(S, g)$.

In this section, we use comultiplication to strengthen the relationship between $\psi(S, \phi)$ and $c(S, \phi)$. Our main result is the following.

Theorem 8.2. *If h and g are two compositions of Dehn twists around curves in S , then there is a comultiplication map*

$$\mu : \widehat{Kh}(S, hg) \rightarrow \widehat{Kh}(S, h) \otimes_{\mathbb{Z}_2} \widehat{Kh}(S, g)$$

which sends $\psi(S, hg)$ to $\psi(S, h) \otimes \psi(S, g)$.

Proof of Theorem 8.2. Suppose that $h = D_{\gamma_1}^{\epsilon_1} \cdots D_{\gamma_p}^{\epsilon_p}$ and $g = D_{\gamma_{p+1}}^{\epsilon_{p+1}} \cdots D_{\gamma_n}^{\epsilon_n}$, and let

$$L_1 = \bigcup_{j=1}^p \gamma_j \times \{t_j\}, \text{ and } L_2 = \bigcup_{j=p+1}^n \gamma_j \times \{t_j\}$$

be the corresponding links in $-M_{S,id}$. Recall that $-M_{S,hg}$ is obtained from $-M_{S,id}$ by performing ϵ_j -surgery on each $\gamma_j \times \{t_j\}$. Let b_1, \dots, b_{2k+r-1} be the arcs on $S = S_{k,r}$ indicated in Figure 26. Choose two points, $0 < T_A < t_1$ and $t_p < T_B < t_{p+1}$, and let β_j be the knot in $-M_{S,id}$ defined by

$$\beta_j = b_j \times \{T_A\} \cup b_j \times \{T_B\} \cup \partial b_j \times [T_A, T_B].$$

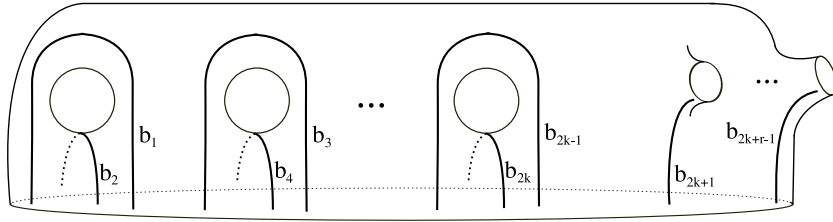


FIGURE 26. The endpoints of the arcs b_j lie in a small collar neighborhood of $\partial S_{k,r}$ which is disjoint from the curves γ_j .

Denote by Y the 3-manifold obtained from $-M_{S,id}$ by performing 0-surgery on each β_j , and let $L_3 \subset Y$ be the induced link

$$L_3 = \bigcup_{j=1}^n \gamma_j \times \{t_j\}.$$

Let $(C(S, hg)_0, D_0)$ be the complex associated to the multi-diagram compatible with all possible combinations of ϵ_j -, 0-, and ∞ -surgeries on the components of L_3 . If $(S, hg)_{i,0}$ is the result of 0-surgery on each induced β_j in $(S, hg)_i$, then

$$C(S, hg)_0 = \bigoplus_{i \in \{0,1\}^n} \widehat{CF}((S, hg)_{i,0}).$$

We define the I -grading on $C(S, hg)_0$ by $I(x) = |i|$ for $x \in \widehat{CF}((S, hg)_{i,0})$. By the construction in [30], there is an I -filtered chain map

$$F : C(S, hg) \rightarrow C(S, hg)_0,$$

which is a sum of maps

$$F_{i,i'} : \widehat{CF}((S, hg)_i) \rightarrow \widehat{CF}((S, hg)_{i,0}),$$

over all pairs i, i' for which $i \leq i'$.

Observe that Y is diffeomorphic to $-M_{S,id} \# -M_{S,id}$. If we think of the links L_1 and L_2 as being contained in the first and second $-M_{S,id}$ summands of Y , respectively (so that there is no linking between L_1 and L_2), then L_3 is simply the union $L_1 \cup L_2$. This becomes evident after a sequence of handleslides (see Figure 27 for an example). In particular,

$$(S, hg)_{i,0} \cong (S, h)_{i_H} \# (S, g)_{i_G},$$

where $i_H = (i_1, \dots, i_p)$ and $i_G = (i_{p+1}, \dots, i_n)$. Therefore, the complex $E_I^1(S, hg)_0$ decomposes as a tensor product,

$$(2) \quad E_I^1(S, hg)_0 \cong E_I^1(S, h) \otimes_{\mathbb{Z}_2} E_I^1(S, g).$$

F induces a map

$$F^1 : E_I^1(S, hg) \rightarrow E_I^1(S, hg)_0$$

which is the sum of the maps

$$(F_{i,i})_* : \widehat{HF}((S, hg)_i) \rightarrow \widehat{HF}((S, hg)_{i,0}),$$

over all $i \in \{0,1\}^n$. Each $(F_{i,i})_*$ is induced by the 2-handle cobordism corresponding to the 0-surgeries on the induced knots $\beta_j \subset (S, hg)_i$. Note that

$$(S, hg)_{i_o} \cong \#^{2k+r-1}(S^1 \times S^2),$$

and the element $\tilde{\psi}(S, hg)$ is the generator of $\wedge^{2k+r-1} H_1((S, hg)_{i_o}) \cong \mathbb{Z}_2$ under the identification of $\widehat{HF}((S, hg)_{i_o})$ with $\wedge^* H_1((S, hg)_{i_o})$. Similarly,

$$(S, hg)_{i_o,0} \cong \#^{4k+2r-2}(S^1 \times S^2),$$

so $\widehat{HF}((S, hg)_{i_o,0})$ may be identified with $\wedge^* H_1((S, hg)_{i_o,0})$. Since the induced knots $\beta_j \subset (S, hg)_{i_o}$ are unknots, $(F_{i_o,i_o})_*$ sends $\tilde{\psi}(S, hg)$ to the generator of $\wedge^{4k+2r-2} H_1((S, hg)_{i_o,0})$, by Proposition 3.2. Under the isomorphism in Equation 2, this generator corresponds to

$$\tilde{\psi}(S, h) \otimes \tilde{\psi}(S, g) \in \widehat{CF}((S, h)_{(i_o)_H}) \otimes \widehat{CF}((S, g)_{(i_o)_G}).$$

The map μ induced on E_I^2 therefore takes $\psi(S, hg)$ to $\psi(S, h) \otimes \psi(S, g)$.

□

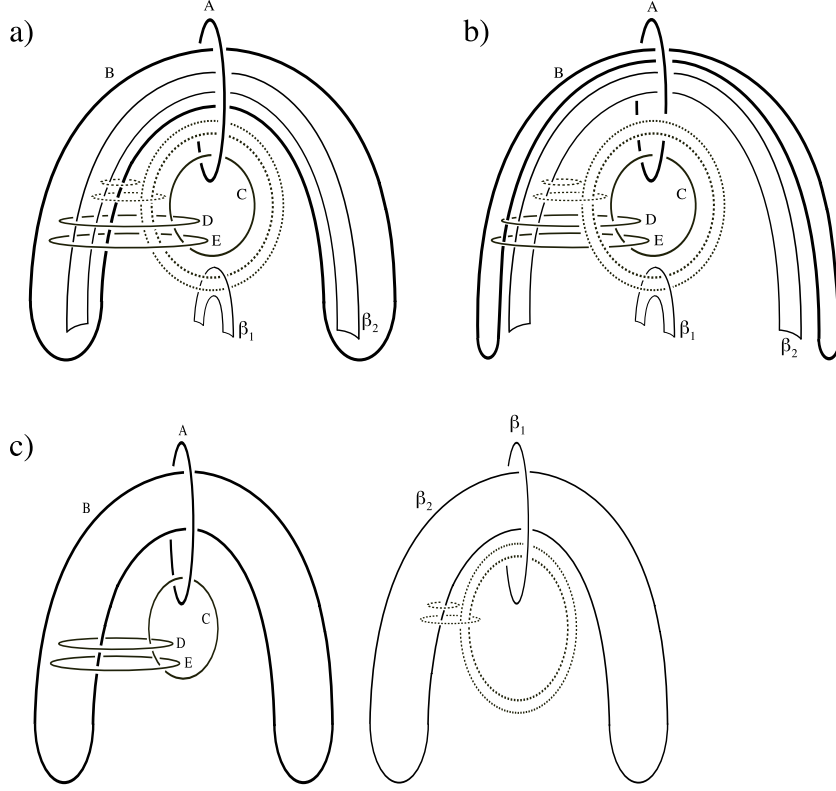


FIGURE 27. In this example, $S = S_{1,1}$, $h = D_{\alpha_1} D_{\alpha_2}^{-2} D_{\alpha_1}^{-1}$, and $g = D_{\alpha_2} D_{\alpha_1}^2$. We begin with the diagram in a). Though we have not written them down, the surgery coefficients on A and B are both 0. The dotted curves comprise the link L_1 corresponding to the diffeomorphism h , and the solid curves C , D , and E comprise the link L_2 corresponding to g . L_3 is the union of L_1 with L_2 , viewed as a link in the 3-manifold Y obtained by performing 0-surgeries on β_1 and β_2 . In Y , slide B over β_2 so that B no longer links the components of L_1 . Next, slide A , D , and E over β_1 so that they no longer link the components of L_1 . We arrive at the diagram in b), which is isotopic to the diagram in c). Note that L_1 and L_2 are unlinked in Y .

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